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THE

HIGHER ARITHMETIC

BY

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AUTHOR OF "LIFE ASSURANCE AND ANNUITY TABLES,
"A NEW GENERAL THEORY OF THE TEETH
OF WHEELS," ETC.

BEING A SEQUEL TO

ELEMENTARY ARITHMETIC

WILLIAM BLACKWOOD AND SONS
EDINBURGH AND LONDON
MDCCCLVII

TO

EDWARD SANG, ESQ.

KIRKCALDY,

THESE PAGES ARE MOST RESPECTFULLY INSCRIBED AS

A TOKEN OF FILIAL AFFECTION.

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PREFACE.

In this Second Volume on Arithmetic an account is given of the doctrines of Powers, Roots, and Logarithms, so far as that can be well done without the aid of general symbols. The Treatise is intended not merely as a Text-Book on these subjects, but also as an introduction to Algebra: indeed, if we adopt the original meaning of the Arab words علم ا لجبر (ylim ul jibr, the science of powers), the present work forms the first, and not the least important chapter of that science.

To those who have only considered the subjects of direct, inverse, and fractional powers, and the cognate subject of Logarithms, in the light which the modern notation throws upon them, it may seem vain to attempt to explain these matters with no aid beyond that of our ordinary numeral notation; but an examination of the following pages may serve to show that the mind does not require the aid of artificial symbols to detect and appreciate even recondite properties of numbers; and the Author flatters himself that he has brought the leading

properties of Logarithms completely within the bounds of arithmetic.

This has been accomplished by the help of a new method for extracting all roots, of which the previously well-known processes for extracting the square and cube roots are the two simplest cases. This method was given, by implication, in a small treatise "*On the Solution of Algebraic Equations of all Orders*, Edinburgh, 1829;" it is here simplified and adapted to ordinary arithmetic. By its means we obtain the root, and all the inferior powers of the root, with great rapidity; the simplicity of the arrangement being the better seen, the higher the order of the root which we extract.

In the actual construction of the first Decimal Logarithmic Tables, Briggs used the repeated extraction of the square root, until the results exceeded unit by fractions so small as to render the excesses sensibly proportional to the exponents. Had he known the method of extracting fifth roots, his labour would have been greatly lessened. The principle used by Briggs is, in essence, identic with that adopted by Dodson in the construction of his Anti-Logarithmic Canon, and with that which is followed at page 119; the only difference is, that the ability to extract fifth roots has given us a much greater command of the subject than either Dodson or Briggs possessed.

The direct computation of the logarithm of a number, that is, in the language of modern algebra, the direct

solution of the equation $a^z = n$, has not heretofore been obtained; for although the well-known formula

$$z = \frac{(n-1) - \frac{1}{2}(n-1)^2 + \frac{1}{3}(n-1)^3 - \frac{1}{4}(n-1)^4 + \text{ etc.}}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^5 + \text{ etc.}}$$

be a symbolical solution, it is only susceptible of direct application when a and n differ from unit by small fractions. In common logarithms $a-1$ has the value 9, and z has to be computed indirectly through the intervention of other numbers.

The student of the Higher Algebra will, therefore, be somewhat surprised to find an exceedingly simple and rapid solution, obtained by a train of reasoning which requires only a clear perception of the nature of powers, and which is altogether independent of notation.

This is another to be added to the rapidly accumulating testimonies of the usefulness of Lord Brouncker's *continued fractions;* for although the algorithm and definition of these fractions have not been employed, the essential idea has been freely used.

These two new processes, viz. the extraction of all roots, and the direct solution of the exponential equation, have enabled the Author to place the whole subject in a clear light, and to complete the Theory of Practical Arithmetic without calling in the dangerous aids of indefinite symbols and arbitrary notation.

In order to prepare the student for following the reasoning to be afterwards used in algebraic investigations, and also for the purpose of fortifying his knowledge of what has already been gone over, a short notice has been added of various Numeration scales. The study of this part of the work may serve to free the mind from those prejudices which are apt to attend the use of a single system, and may lead it to form just and comprehensive views of arithmetic in general.

EDINBURGH, *March* 185

THE HIGHER ARITHMETIC.

In the former volume I have considered at length the ordinary operations of arithmetic. The attentive student cannot have failed to observe that these operations, however simple they may appear to be, really involve important, and, as yet, uninvestigated principles. Those applications which we have made of these principles are easily understood, and the results which we have already obtained lead us to anticipate great advantages from a closer examination of their characters.

The whole system of decimal arithmetic is founded on the use of repeated multiplications by *ten;* one hundred is ten times ten; one thousand is ten times one hundred; and so on. These cardinal numbers are called the *powers of ten;* one hundred is the *second power*, one thousand the *third power*, etc. If, then, a slight knowledge of the properties of the powers of ten have given us such enormous facilities in calculation, what may we not expect from an acquaintance with the properties of the powers of numbers in general?

D. When a number is multiplied by itself, the product again by the original number, and so on, the successive products are called the *powers* of that number; thus, if we multiply 7 by 7, the product 49 again by 7, the new product 343 again by 7, and so on, the resulting numbers 49, 343, 2401, 16 807, etc., are called the powers of seven; just as the numbers 100, 1000,

10 000, etc., are called the powers of ten. I am not sure for what reason this name *power* is given to such products, and it is not very easy to trace any connection between the ordinary meaning of the word *power* and the idea which is here attached to it.

The product obtained by multiplying a number by itself is called the square of that number, because, when we arrange the product in rows, the counters used fill up the figure called a square.

```
* * * * * * *
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* * * * * * *
* * * * * * *
* * * * * * *
```

The product obtained by multiplying the square by the original number is called the *cube*, because, if we were to place as many of these squares, one above another, they would fill up the form to which the name *cube* is given. This word cube comes to us from the French *cube*, the Latin *cubus*, or the Greek *κυβος*: its origin, however, is Arabic. The word ki'ab كعب means anything in the form of a block, and has its plural كعوب kou'oub, or kiu'oub.

The product of the cube by the original number is called the *biquadrate* or second square, because it is also the square of the square.

Further than this, we scarcely use distinctive names, preferring the general term *power* with its ordinal prefix. The number itself is called the *first power;* the square is called the *second power;* the cube is called the *third power.* Then we have the *fourth power*, the *fifth power*, and so on, of the original number.

In this volume I propose to treat of the calculation and properties of such powers, and also of some other subjects closely connected with them.

CHAPTER XVII.

ON SQUARE NUMBERS.

D. THE product obtained on multiplying a number by itself is called the *square* of that number; thus 25 is the square of 5, 289 the square of 17. It is a very easy matter to compute the square of a proposed number.

For the square of a number, for example, of 7, we may write *seven times seven*, with the sign of multiplication, thus 7×7 or 7.7, but it is more usual to write a small ^{2} above and to the right hand of the number which is to be squared, so that 7×7 and 7^2 are different ways of writing 49, the square of 7; the latter form is often read *the second power of* 7, or 7 (*raised*) *to the second power*. The propriety of the notation 7^2 may be best understood when we come to deal with higher powers.

EXAMPLES.

$43^2 =$;	$79^2 =$;	$84^2 =$;
$100^2 =$;	$490^2 =$;	$973^2 =$;
$2795^2 =$,	$5684^2 =$;	$7008^2 =$;
$8001^2 =$;	$9621^2 =$;	$14327^2 =$.

As square numbers are very much used, particularly in calculations connected with geometry and mechanics, it is worth while to compute and to print tables of them, so that we may be spared the labour of computing each one as we need it, and,

in the course of time, of computing the same square over and over again.

In arranging a table of this kind, we prepare a column to receive the numbers, and an adjoining column to receive their squares. Now, if the table were continued to any great length, it would be enormously laborious to calculate each square by multiplication. We do not feel the toil much at the beginning; but, as we come to large numbers, we find the work very toilsome, and, seeking to lessen the labour, we naturally inquire whether the square last found may not be of some use to us in getting the next. For instance, we may have calculated the square of 3573, and found it to be 12 766 329; may not this help us to compute the square of the next number 3574?

Let us examine the matter with smaller numbers. In order to obtain the square of 7 we place 7 rows having seven counters in each row, and, having reckoned them up, we find the square of 7 to be 49. In order to get the square of the next number 8, we need not destroy our previous work; rather let the counters remain, and place 7 new counters down the side and 8 new counters along the bottom:

```
*  *  *  *  *  *  *  o
*  *  *  *  *  *  *  o
*  *  *  *  *  *  *  o
*  *  *  *  *  *  *  o
*  *  *  *  *  *  *  o
*  *  *  *  *  *  *  o
*  *  *  *  *  *  *  o
o  o  o  o  o  o  o  o
```

we shall then have the square of 8; that is to say, by adding 15, the sum of 7 and 8, to the square of 7, we obtain the square of 8.

Similarly, if we had laid out 3573 rows with 3573 counters in each row, we should have had 12 766 329 counters in all. To pass from this to the square of 3574, we must place 3573 counters along the side, and 3574 counters along the bottom; that is to say, we must add $3573 + 3574$ or 7147 to the square of 3573 in order to obtain the square of 3574; and it is quite clear that the same thing holds good of the squares of any other pair of contiguous numbers. A knowledge of

this truth (or *law*, as we call it), enables us to construct a table of squares with great rapidity. Thus,—

3573	12 766 329	3577	12 794 929
	7 147		7 155
3574	12 773 476	3578	12 802 084
	7 149		7 157
3575	12 780 625	3579	12 809 241
	7 151		7 159
3576	12 787 776	3580	12 816 400
	7 153		

the heavy multiplications being superseded by easy additions.

An expert computer saves room by placing the differences in a column alongside of the column of squares, and by adding them as they stand. Almost all extensive tables are computed by help of differences, and therefore the student should accustom himself to add in this way. The calculation would be continued thus,—

Numb.	Square.	Diff.
3580	12 816 400	7 161
3581	12 823 561	7 163
3582	12 830 724	7 165
3583	12 837 889	7 167
3584	12 845 056	
&c.	&c.	&c.

If the student possess a printed table of squares, he can hardly do better than prolong it by a few hundred terms. If he have no table, he may construct one as far as the square of 1000 for himself.

It is clear that the square of a number ending in zero must end in two zeroes, and that, therefore, each tenth square may be checked by comparison with the preceding part of the table; thus the square of 3580 must agree with that of 358. In this way any error in the work is almost sure to be detected.

It is also seen at once that the differences increase by 2 at a time, and that they are all odd; hence the truth of the proposition, "*The sums of the series of odd numbers form the series of square numbers.*"

We have learned how to pass from the square of one number to the square of the next number: let us try to overleap several, and to pass from the square of a number to the square of another number somewhat removed, as from the square of 7 to the square of 10.

Having arranged counters to represent the square of seven, let us place three rows of seven each along the side of the square and we shall have *seven times ten* counters. In order to complete the square of ten, or *ten times ten*, we must now place three rows of ten each along the bottom. So that to the square of 7 we must add three times seven and three times ten, or in all three times seventeen, in order to obtain the square of ten. Now 3 is the difference between the two numbers 7 and 10, while 17 is their sum, so that the difference between the squares of the two numbers 7 and 10 is the product of their sum 17 by their difference 3.

```
* * * * * * * o o o
* * * * * * * o o o
* * * * * * * o o o
* * * * * * * o o o
* * * * * * * o o o
* * * * * * * o o o
* * * * * * * o o o
o o o o o o o o o o
o o o o o o o o o o
o o o o o o o o o o
```

In the same way we can easily satisfy ourselves that *the difference between the squares of any two numbers is the product of the sum of those two numbers by their difference.*

Attention to this circumstance enables us to pass from the square of one number to the square of another. For example, the square of 2817 is 7 935 489, and we wish thence to obtain the square of 2822.

If we had had the counters laid out for the square of 2817,

and wished to complete the square of 2822, we must have placed 5 rows of 2817 counters each, along one side of the square, and then 5 rows of 2822 each along the side of the rectangle; that is to say, we must add 5 times 5639. The computation may be arranged thus :—

	2817	$2817^2 =$	7 935 489
	2822	$5 \times 5639 =$	28 195
Sum, . .	5639	$2822^2 =$	7 963 684
Difference, .	5		

We have, not unfrequently, to compute the difference between the squares of two numbers without needing to know what either of the squares may be: and this we can readily do. Thus if the difference between the squares of 5873 and 6491 be required, we obtain it by multiplying the sum of the two numbers by their difference, thus :—

	5873
	6491
Sum, ...	12364
Difference, . . .	618
	74184
	222552
$6491^2 - 5873^2 =$	7640952

Examples.

The student may verify his results by computing the squares and subtracting.

$8\,245^2 - 8\,171^2 =$; $9\,360^2 - 9\,257^2 =$;

$9\,999^2 - 9\,137^2 =$; $26\,531^2 - 26\,304^2 =$;

$29\,064^2 - 28\,413^2 =$; $35\,868^2 - 29\,982^2 =$;

$57\,264^2 - 57\,160^2 =$; $193\,007^2 - 190\,627^2 =$.

The same matter may be viewed in another light. The larger number 10 is the sum of 7 and 3, and if we place counters to represent its square, we may divide these into four groups, as shown in the margin; one representing the square of 7, another the square of 3, and each of the others the product of 7 by 3; and it is quite clear that the same thing may be done for the square of the sum of any other two numbers. Hence we have this general law, that *the square of the sum of two numbers is made up of the squares of the two numbers together with twice their product.*

```
* * * * * * * o o o
* * * * * * * o o o
* * * * * * * o o o
* * * * * * * o o o
* * * * * * * o o o
* * * * * * * o o o
* * * * * * * o o o
o o o o o o o * * *
o o o o o o o * * *
o o o o o o o * * *
```

An ordinary example in numbers affords a clear illustration of this principle: thus the square of 307 (the sum of 300 and 7) is made up of the square of 300, the square of 7, and two products of 300 by 7, as may be seen on inspecting the details of the multiplication.

```
   307
   307
  2149
 9 21
 9 4249
```

E. From this we have an expeditious method of squaring mentally any number consisting of two figures. For example, the square of 76 is made up of the square of 70, which is 4900, the square of 6, which is 36 (together 4936), and twice 6 times 70, which make 840: the entire square then is 5776. If the two figures be separated by a zero, as in the example 709, the operation is rather easier, since there can never be more than unit carried to the square of the higher digit; while if the figures be separated by more than one zero, the square may be written down at once: thus the square of 9 008 is 81 144 064.

```
 4936
 84
 5776

 490081
 126
 502681

 81000064
 144
 81144064
```

EXAMPLES.

Square mentally the following numbers :—

15	38	79	203	407	1007
19	41	83	208	803	2008
23	49	99	305	909	3009
29	56	101	309	1002	7008
35	67	107	402	1005	

D. This statement that "the square of the sum of two numbers is made up of the sum of their squares and twice their product," does not differ, except in the mere form of words, from the previous statement that "the difference between the squares of two numbers is the product of the sum of those numbers by their difference," as the student may perceive on comparing the one with the other. It is important that we distinguish the essence of a proposition from the appearance of its verbal enunciation.

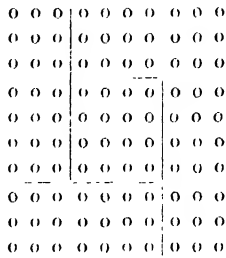
E. The same proposition may be extended to the square of the sum of several numbers, which may be shown to be made up of the squares of the several numbers and twice the product of each pair of them.

Thus 5734 is the sum of 5•••, 7••, 3• and 4, where, for the sake of distinctness, the Arab nokta (•) is used to indicate the rank of the terms, and its square is made up of the squares of the several parts (which may at once be written 25 490 916, since the figures can never interfere with each other) and of twice 5000 times 700, twice 5000 times 30, twice 5000 times 4, twice 700 times 30, twice 700 times 4, and twice 30 times 4, as shown in the accompanying work, and as

5•••²	=	25 ••• •••
7••²	=	49• •••
3•²	=	9••
4²	=	16
2.5••• × 7••	=	7 0•• •••
2.5••• × 3•	=	30• •••
2.5••• × 4	=	40 •••
2. 7•• × 30	=	42 •••
2. 7•• × 4	=	5 6••
2. 3• × 4	=	240
		32 878 756

may be seen on attentively considering the process of squaring by multiplication : in that process any product, such as 5000 by 700, occurs twice, viz. once when multiplying by the 7, and once when multiplying by the 5, while the square 5000 times 5000 only occurs once.

D. The square of the sum of two numbers may be symmetrically divided into five parts, viz. four products of the two numbers at the four corners, and the square of the difference of the two numbers in the centre. Thus 10 is the sum of 7 and 3, while 4 is their difference, and the square of 10 is seen to be made up of four products of 7 by 3, together with the square of 4. This leads us to perceive the general truth that *the square of the sum of two numbers exceeds the square of their difference by four times their product.*



This truth guides us to a beautiful and expeditious method of performing multiplication by means of auxiliary tables. The statement may be put in this form :—

The QUARTER SQUARE *of the sum of two numbers exceeds the* QUARTER SQUARE *of their difference by their* PRODUCT.

If then we have a table of quarter-squares we may readily obtain the product of two numbers. Thus if we desire to multiply 4873 by 3957, we take their sum and their difference ; then having sought out the quarter-squares of these from the table, we subtract the quarter-square of the difference from the quarter-square of the sum : the remainder is the product of the two original numbers.

4 873	
3 957	
8 830	19 492 225
916	209 764
	19 282 461

It often occurs in business that we have to multiply large numbers, and a computer, therefore, seeks by every means to reduce the labour of this operation. The most obvious method

is to prepare a table in which all the products may be entered ; but though we only go as far as to 1000 times 1000, such a multiplication table would fill five hundred quarto pages ; while if we were to go on to 10 000 times 10 000, we should require fifty thousand folio pages. This method then is not practicable.

The advantage of using quarter-squares consists in this, that a very small volume may suffice. Thus M. Antoine Voisin published in 1817 a table of the quarter-squares of all numbers up to 20 000, which though forming only a thin duodecimo, enables us without much trouble to find the product of any two numbers up to 10 000. Mr Laundy has lately published a table of quarter-squares up to 100 000. This work ought to be in the hands of every professional calculator.

In the Appendix I have given a specimen table of quarter-squares up to 2000.

E. In these tables no fractions are shown ; now the square of an odd number is odd, and therefore the quarter of that square must contain a fraction. May not the omission lead us into error ?

Every odd number is the sum of the even number immediately below it and unit ; wherefore the square of every odd number is made up of the square of an even number, twice that even number and unit. Now the square of every even number is divisible by 4, and so is the double of an even number, and therefore when the square of an odd number is divided by 4, there must be *unit* over, and thus each alternate quarter-square must contain the fraction $\frac{1}{4}$.

If the two numbers to be multiplied be both even, their sum and their difference are even also, and the quarter-squares of these have no fraction, as in this example—

876	
518	
1394	485 809
358	32 041
	453 768

If the two numbers be both odd, their sum and their difference are both even, and again there are no fractions ; thus—

961	
823	
1784	795 664
138	4 761
	790 903

Lastly, if one of the numbers be even and the other odd, their sum and their difference must be both odd, and therefore each of the quarter-squares must contain the fraction $\frac{1}{4}$, as in this example.

438	
761	
1199	359400,25
323	26082,25
	333318

In this case the fraction ,25 disappears on subtraction, and thus it seems that in no possible case can the neglect of the fractional parts of the quarter-squares lead to error.

EXAMPLES.

D. Find the products of the following numbers by the method of quarter-squares :—

157 × 194	236 × 218	285 × 249
374 × 288	409 × 567	539 × 723
652 × 934	1005 × 947	1099 × 828
1731 × 259	996 × 982	1176 × 734

D. In many business calculations we have to compute the sum of several products, without needing to care about them individually. For example, a banker wishes to ascertain the interest due on an account-current containing many entries in the course of the year. In the ordinary way he has to multiply each sum of money by the number of the days during which it has been at interest, and to add all these products together. By help of a table of quarter-squares, the work may be carried

on neatly and rapidly, as may be seen from the subjoined example.

£	Days.	Sum.	Diff.	Q. S. Sum.	Q. S. Diff.
528	321	849	207	18 0200	1 0712
657	307	964	350	23 2324	3 0625
475	263	738	212	13 6161	1 1236
212	241	453	29	5 1302	210
419	179	598	240	8 9401	1 4400
584	117	701	467	12 2850	5 4522
237	93	330	144	2 7225	5184
684	69	753	615	14 1752	9 4556
				98 1215	22 1445
				22 1445	
		Sum of Products,	=	75 9770	

By this operation we obtain the sum of all the products at once; and it is no small recommendation to the process that every figure of the work is put upon record, so that an error may be traced to its source.

Example.

A merchant's deposit-account with a bank was as under :—

1854.			
Jan.	2.	By balance from 1853,	£973
,,	12.	To cash withdrawn,	159
Feb.	17.	To cash withdrawn,	465
Mar.	10.	By cash deposited,	368
April	7.	By cash deposited,	183
May	23.	To cash withdrawn,	762
June	9.	By cash deposited,	497
July	24.	By cash deposited,	217
Aug.	10.	To cash withdrawn,	678
Sept.	29.	To cash withdrawn,	187
Oct.	4.	By cash deposited,	373
Nov.	14.	By cash deposited,	198
Dec.	29.	To cash withdrawn,	210

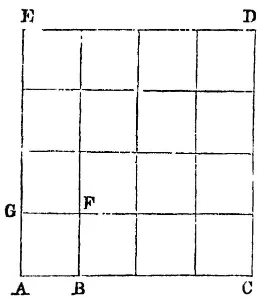
Required the state of the account as on January 1, 1855, the interest allowed being at the rate of 3 per cent per annum.

D. Hitherto I have considered the squares of whole numbers, and now proceed to treat of the squares of fractions.

In order to find the product of two fractions we multiply the numerators together and the denominators together; so that, in order to square a fraction, we must square the numerator and square the denominator. In this way we find that the square of *one-half* is *one-quarter*.

Many a beginner is startled at such a statement. Now, it is essential to his further progress that he thoroughly understand this matter, and therefore I shall enter somewhat minutely into the consideration of it.

The *multiplication*, as we call it, of one fractional number by another occurs in a variety of business calculations, and it is only in reference to such calculations that the operation has any intelligible meaning. Thus, if we wish to compute the surface of a rectangular board, we multiply the number of the linear units in the length by that of the linear units in the breadth, in order to obtain the number of square units contained in the surface; and when the length and breadth are represented by fractional numbers, we *multiply* the one of those fractional numbers by the other. If, then, we have to compute the surface of a square board, we have to *multiply* that number which represents the side of the board by itself.



Thus, a square of which the side is 4 inches, contains 16 square inches; while the square on the fourth part of an inch contains only the 16th part of a square inch. If we suppose that AB, in the adjoining figure, is 1 inch, AC is then 4 inches, and the square ACDE contains the square inch ABFG four times four, that is sixteen times. But if we suppose AC to be 1 inch, then AB must be one quarter of an inch, and its

square the sixteenth part of a square inch. Hence the statement that the square on 4 inches contains 16 square inches, is just the same as this other statement, that the square of $\frac{1}{4}$ of an inch is $\frac{1}{16}$ of a square inch.

It is also very easy to see that the square on three-quarters of an inch contains nine-sixteenths of the square inch.

The subject may also be viewed in this light. Five yards of cloth, at five shillings the yard, cost twenty-five shillings. Three yards of cloth, at three shillings the yard, cost nine shillings. One yard, at one shilling per yard, costs one shilling; and half a yard of cloth, at half a shilling per yard, costs one quarter of a shilling.

E. Fractions occur when we come to apply arithmetic to business or to other departments of science; they belong to applicate rather than to pure arithmetic, and hence it is that we can only understand the nature of the operations into which they enter by studying those practical questions which give rise to them. We cannot explain the subject without having recourse to illustrations borrowed from other sciences. I have given, above, one illustration borrowed from geometry, and another taken from mercantile affairs, and shall now offer, to the somewhat advanced student, yet another illustration from the science of motion.

When a heavy body is allowed to fall, it moves more and more rapidly as it descends. The law according to which it moves has been carefully examined, and it has been found that in the first second of time a stone falls rather more than 16 English feet; in the next second it falls *three* times as far; in the third second *five* times; in the fourth second *seven* times as far, and so on; so that in two seconds a stone falls four times; in three seconds, nine times; in four seconds, sixteen times as far as in one second, and so on, the distances being proportional to the squares of the numbers of seconds during which it has fallen.

This being the law of motion of a falling body, it follows that in one-half of a second the distance is one-fourth part of

16 feet; in one-third of a second the distance is one-ninth part of 16 feet; and in one fourth part of a second the distance is one-sixteenth part of the 16 feet which a body falls in one second.

If, then, we wish to calculate how far a stone falls in a period of time indicated by fractions of a second, we must square the fraction in the manner which has been explained. Thus, if we desire to know how far a stone falls in $2\frac{1}{3}$ seconds of time, we consider that this is seven third-parts of a second. Now, in one-third of a second the stone falls one-ninth part of 16 feet, and in seven-thirds it falls forty-nine times as far as in one-third of a second; wherefore in $2\frac{1}{3}$ seconds the stone falls $\frac{49}{9}$ of 16 feet.

D. EXAMPLES.

Square the following fractions:—

$\frac{2}{3}$	$\frac{9}{13}$	$\frac{24}{29}$	$\frac{92}{701}$	$\frac{1}{173}$
$\frac{3}{5}$	$\frac{5}{19}$	$\frac{17}{38}$	$\frac{108}{299}$	$\frac{743}{964}$
$\frac{5}{7}$	$\frac{1}{10}$	$\frac{41}{51}$	$\frac{52}{469}$	

D. When we have to square a mixed number, as $4\frac{1}{3}$, we may convert it into the single fraction $\frac{13}{3}$; squaring this we find $\frac{169}{9}$, which may be put under the form $18\frac{7}{9}$.

Otherwise we may observe that the square of the sum of two numbers is made up of the squares of those numbers and twice their product, whether the numbers be fractional or integer. Thus the square of $4\frac{1}{3}$ is made up of the square of 4, viz. 16; the square of $\frac{1}{3}$, viz. $\frac{1}{9}$; and twice the product of 4 by $\frac{1}{3}$, viz. $\frac{8}{3}$ or $2\frac{2}{3}$; that is, in all, $16 + \frac{1}{9} + 2\frac{2}{3}$, or $18\frac{7}{9}$.

EXAMPLES.

Square the following mixed numbers by both of the above processes:—

$3\frac{2}{7}$	$164\frac{3}{82}$	$14\frac{3}{7}$	$20\frac{3}{5}$	$359\frac{13}{17}$
$8\frac{9}{17}$	$10\frac{1}{5}$	$159\frac{119}{143}$	$75\frac{7}{15}$	$258\frac{17}{43}$

When we have to square a decimal fraction we proceed exactly as when multiplying one decimal fraction by another.

EXAMPLES.

Square the following fractions :—

,501	3,0053	1,000962	87,079	,0387
,9735	,0002916	357,803	,100572	3,00628

It is worthy of remark that the square of the product of two numbers is the product of the squares of those numbers ; thus, 15 is the product of 3×5, and 15^2 is the product of 9 by 25.

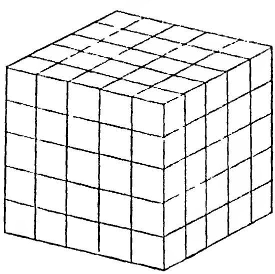
CHAPTER XVIII.

ON CUBE NUMBERS.

D. THE product of the square of a number by the number itself is called the *cube* of that number. Thus if 25, the square of 5, be multiplied by 5, the product, 125, is called the cube of 5.

As the square of a number is indicated by a small exponent ^{2}, so the cube is indicated by a small ^{3}; thus 5^3 means the cube, otherwise called the *third power* of 5, the ^{3} meaning that there are three equal factors; so that 125, 5^3 and $5 \times 5 \times 5$ are different ways of writing the same thing. If, having arranged counters to represent the square of 5, as in the margin, we were to pile above it, at equal distances, other four such squares, we should have 5 times the square of 5 arranged in the form of a cube. Such an arrangement may be best shown on paper, by making each counter of the form of a cube or die.

*	*	*	*	*
*	*	*	*	*
*	*	*	*	*
*	*	*	*	*
*	*	*	*	*



It is then no difficult matter to compute the cube of any number; and if we possess a table of squares much of the work is saved to us.

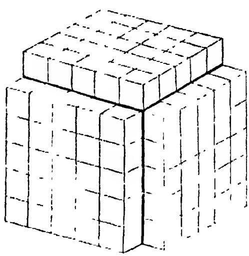
Examples.

Cube the following numbers :—

3	78	695	2816
7	100	828	3907
15	137	901	5483
26	199	1573	9983
55	271	1794	

Cube numbers do not occur in business nearly so often as square numbers, yet they occur so frequently that it is desirable to have tables of them prepared, the more so that the labour of cubing a large number is great. The earlier part of a table of cubes may be computed by actual multiplication ; but it would be altogether out of the question to go on far in this way. Now by examining the difference between one square number and the next, we discovered a very easy method of computing a table of squares : let us try whether we cannot pass from the cube of one number to the cube of the next.

Having laid together as many dies as make up the cube of 5, we may, in order to pass to the cube of 6, place on each of the three faces of the cube a square of 5, as shown in the figure: and then it is clear that we must fill up the hollows left along the three edges by three rows of 5 each ; this would still leave room for a single die at the corner. Hence it appears that the cube of 6 is



made up of the cube of 5, 3 squares of 5. 3 times 5, and unit. In the same way the cube of 7 exceeds the cube of 6 by 3 squares of 6, 3 times 6, and unit, or, as it may be written—

$$7^3 = 6^3 + 3.6^2 + 3.6 + 1$$

product of the square of 5 by 2, three times the product of the square of 2 by 5, and the cube of 2 ; or as we may write it—

$$7^3 = 5^3 + 3.5^2.2 + 3.5.2^2 + 2^3,$$

and a similar statement may be made for any other numbers ; for example, 17 is the sum of 12 and 5, and the cube of 17 is made up thus—

$$17^3 = 12^3 + 3.12^2.5 + 3.12.5^2 + 5^3,$$

and this statement may be put in words thus—

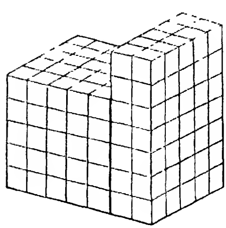
The cube of the sum of two numbers is made up of the cubes of those numbers and three times the products of each number by the square of the other.

E. The truth of this statement may be illustrated without any appeal to the geometrical arrangement of the cube. Let it be proposed to cube a number of two digits, such as 73. This number is the sum of two parts, 7· and 3. Now on effecting at once the continued multiplication of the three factors 73 × 73 × 73 in the manner explained in Chap. IV., p. 88, we have first the cube of 7·, then we have the square of 7· multiplied by 3 repeated thrice, next the square of 3 multiplied by 70 repeated thrice, and lastly the cube of 3 ; so that—

$$\begin{array}{r} 73 \\ 73 \\ 73 \\ \hline 343\cdot\cdot\cdot \\ 147\cdot\cdot \\ 147\cdot\cdot \\ 147\cdot\cdot \\ 63\cdot \\ 63\cdot \\ 63\cdot \\ 27 \\ \hline 389017 \end{array}$$

$$73^3 = 70^3 + 3.70^2.3 + 3.70.3^2 + 3^3.$$

D. This composition of the cube of the sum of two numbers may be otherwise considered. Thus on one face of the cube of 5 let us lay a block 7 long, 5 broad, and two thick. When this block is laid up to one edge, as shown in the figure, its other edge projects 2 beyond the line of the cube, and if on each of the two adjoining faces another such block



were placed, these would build up the cube of 7, wanting the

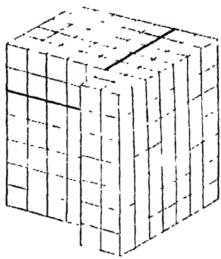
cube of 2 at the corner. Hence the cube of 7 is made up of the cubes of 5 and of 2, and of three times the continued product of 7, 5, and 2, or—

$$7^3 = 5^3 + 3.7.5.2 + 2^3\ ;$$

and similarly, 17 being the sum of 12 and 5—

$$17^3 = 12^3 + 3.17.12.5 + 5^3.$$

There is still a third way in which the difference between two cubes may be divided. For example, the difference between the cube of 5 and the cube of 7 may be obtained thus: on the top of the cube of 5 let us lay two squares of 5, and at the one side let us set up two squares of 7, as shown in the figure, then do we need, in order to make up the cube of 7, two products of 5 by 7. And thus we see that the



difference between the cube of 5 and the cube of 7 is made up of twice the square of 5, twice the square of 7, and twice the product of 5 by 7, or is the product of the sum of the squares of the two numbers and their product by the difference of the two numbers; or—

$$7^3 - 5^3 = (7^2 + 5^2 + 7.5)\ 2.$$

In the very same way—

$$17^3 - 12^3 = (17^2 + 12^2 + 17.12)\ 5.$$

Examples.

Compute by each of the three methods the following differences of cubes:—

$19^3 - 13^3 =$;	$27^3 - 17^3 =$;
$58^3 - 42^3 =$;	$63^3 - 23^3$;
$78^3 - 22^3$;	$81^3 - 77^3$;
$85^3 - 79^3$;	$99^3 - 91^3$.

The cube of a fraction has for its numerator the cube of the numerator, and for its denominator the cube of the denominator of the fraction. Thus—

$$(\tfrac{3}{5})^3 = \tfrac{3}{5} \times \tfrac{3}{5} \times \tfrac{3}{5} = \tfrac{27}{125}.$$

In the same way we find that the cube of one-half is one-eighth, and the cube of one-third is one twenty-seventh. It is easy to satisfy ourselves that this ought to be the case, for since one yard is three feet, one cubic yard must contain twenty-seven cubic feet; and therefore, since one foot is the third part of a yard, one cubic foot must be the twenty-seventh part of a cubic yard; and thus, the statement that the cube of three is twenty-seven is identic with the statement that the cube of one-third is one twenty-seventh.

Examples.

Cube the following fractions :—

$\frac{1}{4}$	$\frac{9}{10}$	$\frac{25}{27}$	$\frac{3}{158}$
$\frac{2}{3}$	$\frac{14}{15}$	$\frac{46}{47}$	$\frac{192}{195}$
$\frac{1}{5}$	$\frac{16}{15}$	$\frac{95}{101}$	$\frac{569}{928}$
$\frac{3}{4}$	$\frac{17}{21}$	$\frac{97}{142}$	$\frac{753}{964}$

When we have to cube a mixed number, the most convenient process is to convert it into a single fraction, and to re-convert the cube of that into a mixed number. Thus, in order to find the cube of $2\frac{1}{4}$, we change it into $\frac{9}{4}$, of which the cube is $\frac{729}{64}$, or $11\frac{25}{64}$.

But the cube may also be found by regarding $2\frac{1}{4}$ as the sum of 2 and $\frac{1}{4}$, whence

$$\begin{aligned}(2\tfrac{1}{4})^3 &= 2^3 + 3.\,2^2.\tfrac{1}{4} + 3.\,2.\,(\tfrac{1}{4})^2 + (\tfrac{1}{4})^3 \\ &= 8 \quad + 3 \qquad + \tfrac{3}{8} \qquad + \tfrac{1}{64} \\ &= 11\tfrac{25}{64}\end{aligned}$$

Examples.

Cube the following fractional numbers :—

$1\frac{1}{2}$	$9\frac{1}{3}$	$38\frac{3}{22}$	$54\frac{2}{3}$	$328\frac{1}{13}$
$3\frac{2}{7}$	$15\frac{7}{15}$	$99\frac{7}{9}$	$207\frac{29}{57}$	$867\frac{13}{17}$

When we have to cube a decimal fraction, we proceed exactly as in the multiplication of decimals.

EXAMPLES.

Cube the following decimals.—

3,5	,999
4,7	,995
19.31	89,207
11,10	1,30578
1,03	5,00019
1,001	,000975
1,005	2,00377

The properties of square and cube numbers derive great importance from their relation to matters of ordinary business, and particularly from their analogy to the surfaces and masses of bodies.

If two maps of the same country be made on different scales, the extents of their surfaces are not proportional to the magnitudes of the scales, but to the squares of those magnitudes; thus, to double the scale of a map is to enlarge the surface fourfold; and to magnify the scale 10 times is to spread the map over 100 times as much paper.

The volumes of bodies of the same shape are proportional to the cubes of their dimensions: thus, if two models of the same group of statuary have their heights in the ratio of 2 to 3, their surfaces are in the ratio of 4 to 9, but their bulks are in the still higher ratio of 8 to 27. Again, if one ball have its diameter five times that of another, its surface is 25 times the surface, and its mass 125 times the mass of that other. For example, the diameter of the sun is about 110 times the diameter of the earth; its surface is then 12 100 times the surface, and its bulk is 1 331 000 times the bulk of the earth.

A clear perception of these truths is of the utmost importance to the engineer.

CHAPTER XIX.

ON THE HIGHER POWERS OF NUMBERS.

D. The product of the cube or third power of a number by the number itself is called its *Fourth power*, or *biquadrate*, which is, therefore, the continued product of four equal factors. The fourth power of 7 is 2401, the continued product of 7.7.7.7; it is, then, appropriately denoted by the expression 7^4 where the exponent ^{4} shows the number of the factors.

Examples.

Find the fourth powers of the following numbers:—

3	18	94	997
5	27	101	1001
8	51	159	5711
10	68	760	8946
13	87	808	59498

$2\frac{1}{2}$	$6\frac{1}{3}$	$\frac{1}{2}$	$\frac{3}{7}$	$\frac{152}{153}$
$1\frac{1}{10}$	$19\frac{1}{11}$	$\frac{1}{5}$	$\frac{1}{47}$	$\frac{101}{100}$
$4\frac{2}{3}$	$\frac{5}{17}$	$\frac{1}{10}$	$\frac{57}{64}$	

1,3	,152	25,1568	,000569
3,05	17,64	1,001	390,99785
7,941	,0175	99,004	584,1928

Instead of obtaining the fourth power by the continued mul-

tiplication of the four factors, we may obtain it by squaring the square; thus 2401, the fourth power of 7, is the square of 49.

Fourth powers are not needed so often as to make it worth while to compute and print tables of them; yet they occur often enough to render an acquaintance with their general properties desirable.

If we compute the fourth powers of a few numbers in order, and take their differences, the differences of these differences, and so on, we find that the differences of the fourth class are all alike; just as we found that the third differences of cube numbers are all alike. This is seen in the following example:—

Num.	4th Power.	1st Diff.	2d Diff.	3d Diff.	4th Diff.
1	1	15	50	60	24
2	16	65	110	84	24
3	81	175	194	108	24
4	256	369	302	132	24
5	625	671	434	156	24
6	1296	1105	590	180	24
7	2401	1695	770	204	
8	4096	2465	974		
9	6561	3439			
10	10000				

And thus a table of fourth powers may be written out without enormous labour.

Example.

Make a table of the fourth powers of all numbers up to 40.

E. We have learned how to pass from the square of one number to the square of the next number, and from one cube to the next cube; let us now try to pass from one biquadrate to another. Here the aid which we obtained from geometrical

arrangement ceases, and we must look out for some other help. We might employ, in this matter, the notation of modern algebra; but I judge it better that we should trust to our own mental exertions rather than that we should depend for our knowledge of principles on the manipulation of symbols.

When we proceed to square mentally such a number as 302, or to multiply it by itself as in the margin, we readily see that the product is made up of the square of 300, two products of 300 by 2, and the square of 2.

```
    302
    302
 ------
 9 •• ••
    6 ••
    6 ••
       4
 ------
 91 204
```

Again, when we multiply at once the three numbers, 302, 302, 302, so as to get the cube of 302, we find the result composed of the cube of 300, three products of the square of 300 by 2, three products of 300 by the square of 2, and the cube of 2.

This may be clearly seen on studying the continued multiplication of three such numbers as 207, 103, 405, performed according to the method explained in Chapter IV., p. 82, and by supposing that these numbers become all equal to each other. In this way we see that the cube of the sum of two numbers is made up of the cubes of those numbers, and of three times the product of each number by the square of the other, as we have already found.

```
        3 02
        3 02
        3 02
 -----------
 27 •• •• ••
    18 •• ••
    18 •• ••
    18 •• ••
       12 ••
       12 ••
       12 ••
           8
 -----------
 27 54 36 08
```

The same method of examination may be applied to biquadrates.

If we have to take the continued product of four factors, each consisting of two parts, as in the margin, in which each factor is composed of *hundreds* and of *units;* we

```
 207
 103
 405
 608
 ---
```

take first the continued product of all the hundreds, then the continued product of each three of the hundreds by the units of the remaining number, there being *four* of these products; then the products of each pair of hundreds by the pair of units in the other numbers, there being *six* of these products; then the products of each one of the hundreds by the units of the other three numbers, there being *four* of these; and lastly, the continued product of the four units.

```
48 •• •• •• ••
   64 •• •• ••
   60 •• •• ••
 1 44 •• •• ••
 1 68 •• •• ••
      80 •• ••
    1 92 •• ••
    1 80 •• ••
    2 24 •• ••
    2 10 •• ••
    5 04 •• ••
       2 40 ••
       2 80 ••
       6 72 ••
       6 30 ••
          8 40
--------------
52 50 08 30 40
```

Let us now suppose that the four factors are all alike, each of them being, say, 302; then in computing the fourth power of 302 we shall have, in the first place, the fourth power of 300; in the second place, four products of the third power of 300 by 2; in the third place, six products of the square of 300 by the square of 2; fourthly, 4 products of 300 by the cube of 2; and lastly, the fourth power of 2: and thus:—

```
          3 02
          3 02
          3 02
          3 02
--------------
81 •• •• •• ••
   54 •• •• ••
   54 •• •• ••
   54 •• •• ••
   54 •• •• ••
      36 •• ••
      36 •• ••
      36 •• ••
      36 •• ••
      36 •• ••
      36 •• ••
         24 ••
         24 ••
         24 ••
         24 ••
            16
--------------
83 18 16 96 16
```

$$302^4 = 300^4 + 4.300^3.2 + 6.300^2.2^2 + 4.300.2^3 + 2^4.$$

It is quite obvious that, although we have made use of the decimal notation to keep the two parts of the number distinct from each other, the same kind of reasoning might be applied to show that *the fourth power of any two numbers is made up of the fourth powers of those numbers, four times the product of the cube of each number by the other, and six times the product of the square of the one by the square of the other number.*

F. For example, in order to trace the composition of the fourth power of *thirteen* considered as the sum of *nine* and *four*, we may regard the abacus as arranged according to the powers of nine, instead of, as usual, according to the powers of ten; and then the number *thirteen* would be represented as in the margin, where the Indian numerals are used to avoid the tedious repetition of counters. The multiplication of the four equal factors, as it would be performed on the nonary abacus, is also shown, and exactly as before the fourth power of *thirteen* is seen to be composed of the fourth powers of nine and four, of four times the product of the cube of nine by four, of four times the product of nine by the cube of four, and of six times the product of the square of nine by the square of four; or, as it may be written symbolically, that,— $$13^4 = 9^4 + 4.9^3.4 + 6.9^2.4^2 + 4.9.4^3 + 4^4.$$

			1	4
			1	4
			1	4
			1	4
1	•	•	•	•
	4	•	•	•
	4	•	•	•
	4	•	•	•
	4	•	•	•
		16	•	•
		16	•	•
		16	•	•
		16	•	•
		16	•	•
		16	•	•
			64	•
			64	•
			64	•
			64	•
				256

D. If then we have to pass from the fourth power of one number to the fourth power of the next number, we must add four times the *cube* of the first number, six times its square,

four times the number and unit; thus to pass from the fourth power of 13 to that of 14, we observe that

$$\begin{aligned} 14^4 &= 13^4 + 4.13^3.1 + 6.13^2.1^2 + 4.13.1^3 + 1^4 \\ &= 13^4 + 4.13^3 \quad + 6.13^2 \quad\quad + 4.13 \quad\quad + 1 \end{aligned}$$

whence the calculation,—

$$\begin{array}{lcr}
13^4 & = & 28561 \\
4.13^3 & = & \left.\begin{array}{r} 8788 \\ 1014 \\ 52 \\ 1 \end{array}\right\} 9855 \\
\hline
14^4 & = & 38416 \\
4.14^3 & = & \left.\begin{array}{r} 10976 \\ 1176 \\ 56 \\ 1 \end{array}\right\} 12209 \\
\hline
15^4 & = & 50625 \\
4.15^3 & = & \left.\begin{array}{r} 13500 \\ 1350 \\ 60 \\ 1 \end{array}\right\} 14911 \\
\hline
16^4 & = & 65536 \\
4.16^3 & = & \left.\begin{array}{r} 16384 \\ 1536 \\ 64 \\ 1 \end{array}\right\} 17985 \\
\hline
17^4 & = & 82521
\end{array}$$

$$\begin{array}{ll} 6.13^2 = & 6.14^2 = \quad 6.15^2 = \quad 6.16^2 = \\ 4.13 = & 4.14 = \quad 4.15 = \quad 4.16 = \\ 1 = & 1 = \quad 1 = \quad 1 = \end{array}$$

Now, although this be a very tedious calculation, so tedious that no one would think of using it, it is most instructive, for it enables us to perceive clearly the nature of the more convenient process by means of successive differences.

The difference between the fourth power of 13 and the fourth power of 14 is 9855; the difference between the fourth power of 14 and that of 15 is 12209, and so on, as shown at the side. Let us endeavour to trace the manner in which these differences grow.

The first of them is made up of 4 times the cube of 13, 6 times its square, 4 times 13 itself, and unit.

The next of them is made up of 4 times the cube of 14, 6 times its square, 4 times 14 itself, and unit.

And so on.

Therefore the difference between the first and second of these differences must be 4 times the difference between the cube of 13 and the cube of 14, 6 times the difference between the squares of the same numbers and 4 times unit (the difference between 13 and 14).

Now, according to what has been shown in the preceding chapters, the difference between the cube of 13 and the cube of 14 is composed of 3 times the square of 13, 3 times 13 and unit; while the difference between the square of 13 and that of 14 is twice 13 and unit: wherefore the difference between the first and second of our differences must be made up thus,—

$$4\{3.13^2+3.13+1\}+6\{2.13+1\}+4;$$

or thus, $$12.13^2+24.13+14.$$

This difference of the differences is the first of what we call the second-differences: and the next of these second-differences would clearly be derived from the number 14 just as this one is from 13, it would be

$$12.14^2+24.14+14,$$

and the next of the second-differences would be

$$12.15^2+24.15+14;$$

and so on.

It is easy to see that the difference between the first and second of these second-differences must be

$$25.13+24,$$

which is the first of the third-differences. The second of the third-differences must be

$$24.14+24,$$

and therefore the first of the fourth-differences must be 24.

Hence if we form a table of fourth powers by the method of successive differences, we must, when we have arrived at the number 13, find the line of differences as under.

Biquadrate,	$13^4 =$	28561
1st Difference,	$4.13^3+6.13^2+4.13+1 =$	9855
2d Difference,	$12.13^2+24.13+14 =$	2354
3d Difference,	$24.13+36 =$	348
4th Difference,	$24 =$	24

So that the construction of a table of biquadrates, beginning from 13, may be carried on by successive additions, as under.

Num.	4th Power.	1st Diff.	2d Diff.	3d Diff.	4th Diff.
13	28561	9855	2354	348	24
14	38416	12209	2702	372	24
15	50625	14911	3074	336	24
&c.	&c.	&c.	&c.	&c.	&c.

EXAMPLE.

Construct a table of fourth powers, beginning at 100.

E. From this investigation it appears that the fourth-difference of the fourth powers must be four times the third-difference of the third powers: now that was seen to be thrice the second-difference of the second powers, which again is twice unit: therefore the fourth-difference of the biquadrates of the series of natural numbers must be the continued product of 1, 2, 3, and 4, or $1 \times 2 \times 3 \times 4$. And we shall see that the same law continues among powers still higher.

D. On raising any of the numbers 11, 1·1, 1··1, 1···1, etc., to the second, third, and fourth powers, we obtain types of the manner in which the power of a sum is composed: thus—

$$11^2 = 121, \quad 1{\cdot}1^2 = 1{\cdot}2{\cdot}1, \text{ \&c.},$$

are obvious exemplifications of the manner in which the square of the sum of two numbers is composed;

$$11^3 = 1331; \quad 1{\cdot}1^3 = 1{\cdot}3{\cdot}3{\cdot}1, \text{ \&c.},$$

exemplify the composition of the cube of the sum of two numbers; and

$$11^4 = 14641; \quad 1{\cdot}1^4 = 1{\cdot}4{\cdot}6{\cdot}4{\cdot}1, \text{ \&c.},$$

exhibit the composition of the fourth power.

But if we continue the same process for higher powers, until the digits in any one rank interfere with those of the next rank, the analogy is destroyed. Hence it is convenient to use the numbers 101, or 1001, for the higher powers; 11 fails us at the fifth power.

D. The fifth power of a number is got by multiplying the fourth power by the number itself; thus on multiplying 81, which is the fourth power of 3, by 3, we obtain 243, the fifth power of 3. This is usually written 3^5, the exponent ^{5} indicating that the fifth power is the product of five equal factors: thus,—

$$243 = 3.3.3.3.3$$

EXAMPLES.

Compute the fifth powers of the following numbers:—

4	11	$\frac{7}{11}$	1·72	·231
7	$17\frac{1}{3}$	$12\frac{1}{8}$	1·004	·0897
$3\frac{1}{2}$	$\frac{3}{4}$	$\frac{99}{100}$	2·734	·00217

When we make a table of the fifth powers of the successive numbers, take their differences, the differences of these, and so on, we find that the differences of the fifth order are all alike, each being 120; and it is to be remarked that this number 120 is five times the fourth difference of the fourth powers, or the continued product of the factors 1, 2, 3, 4, 5. This circumstance enables us readily to continue the table.

EXERCISE.

Construct a table of the fifth powers of all numbers from 40 to 70 by the method of differences.

In order to discover the difference between the fifth powers of two numbers, or what comes to the same thing, to discover how the fifth power of the sum of two numbers is made up, we may study the continued product of five numbers, such as 2·7, 1·3, 4·5, 6·8, and 3·7, and then consider what would result from supposing all these numbers to become alike: or we may rest contented with computing the fifth power of some number, such as 1·1, 1··1, &c. The fifth power of 1·1 is 1 ·5 10 10 ·5 ·1, that of 1001 is 1 ··5 ·10 ·10 ··5 ··1. Grouping the first of these in 2's, and the second in 3's, we con-

clude that *the fifth power of the sum of two numbers is composed of once the fifth power of each number, five times the product of the fourth power of each into the other number, and ten times the product of the cube of each into the square of the other.* This may be still more clearly seen by examining the fifth power of 1 • • 2, which is

1 •10 •40 •80 •80 •32,

and is composed of 1 ••• ••• ••• ••• •••, the fifth power of 1000, 10 ••• ••• ••• •••, five times the fourth power of 1000 multiplied by 2, 40 ••• ••• •••, ten times the third power of 1000 multiplied by the square of 2, 80 ••• •••, ten times the square of 1000 multiplied by the cube of 2, 80 •••, five times 1000 multiplied by the fourth power of 2, and of 32, the fifth power of 2 : thus

$$1002^5 = 1000^5 + 5.1000^4.2 + 10.1000^3.2^2$$
$$+ 10.1000^2.2^3 + 5.1000.2^4 + 2^5.$$

In the same way 19 being the sum of 12 and 7, we have—

$$19^5 = 12^5 + 5.12^4.7 + 10.12^3.7^2 + 10.12^2.7^3 + 5.12.7^4 + 7^5 .$$

If then we desire to pass from the fifth power of one number to the fifth power of the next number, as from the fifth power of 12 to that of 13, we have only to observe that 13 is the sum of 12 and of 1, and that all the powers of unit are unit. Hence

$$13^5 = 12^5 + 5.12^4 + 10.12^3 + 10.12^2 + 5.12 + 1 .$$

The difference between the fifth power of 12 and that of 13 is thus

$$5.12^4 + 10.12^3 + 10.12^2 + 5.12 + 1,$$

and the next difference would be

$$5.13^4 + 10.13^3 + 10.13^2 + 5.13 + 1,$$

and so on ; wherefore the first of the second-differences, being the difference between these, must be thus composed :—

$$5\{4.12^3 + 6.12^2 + 4.12 + 1\}$$
$$10\{3.12^2 + 3.12 + 1\}$$
$$10\{2.12 + 1\}$$
$$5\{1\}$$

making in all

$$20.12^3 + 60.12^2 + 70.12 + 30 .$$

The next of the second-differences would necessarily be

$$20.13^3 + 60.13^2 + 70.13 + 30\,;$$

wherefore the first of the third differences must be

$$\begin{array}{r} 20\{3.12^2 + 3.12 + 1\} \\ 60\{2.12 + 1\} \\ 70\{1\} \end{array}$$

making in all

$$60.12^2 + 180.12 + 150.$$

Of course the next of the third-differences would be

$$60.13^2 + 180.13 + 150,$$

so that the first of the fourth-differences must be

$$\begin{array}{r} 60\{2.13 + 1\} \\ 180\{1\} \end{array} \quad \text{or—}$$

$$120.12 + 240\,;$$

and the next of the fourth-differences

$$120.13 + 240\,;$$

whence it is clear that the difference of the fifth order must be 120.

While computing a table of fifth powers by the method of successive differences, the numbers in the line opposite 12 must be—

Fifth power	=	12^5
1st Difference	=	$5.12^4 + 10.12^3 + 10.12^2 + 5.12 + 1$
2d Difference	=	$20.12^3 + 60.12^2 + 70.12 + 30$
3d Difference	=	$60.12^2 + 180.12 + 150$
4th Difference	=	$120.12 + 240$
5th Difference	=	120

that is—

Fifth power	=	248 832
1st Difference	=	122 461
2d Difference	=	44 070
3d Difference	=	10 950
4th Difference	=	1 680
5th Difference	=	120

E. It would be endless to go on examining the sixth, seventh, and subsequent powers in this way. The student who desires

to make himself thoroughly master of the subject, may pursue the same line of investigation one or two steps farther, and may satisfy himself that the sixth-difference of the sixth powers of the natural numbers is the product 1.2.3.4.5.6 ; and analogously of the seventh power.

D. Having now studied sufficiently for our present purpose the relations of the same powers of different numbers, we may proceed to inquire into the relations of different powers of the same number.

In order to form a list of the successive powers of a given number, we have only to multiply repeatedly by that number ; thus—

3	$=3$	27	$=3^3$	243	$=3^5$
9	$=3^2$	81	$=3^4$	&c.	&c.

EXAMPLE.

Form a table of the ten first powers of the ten first numbers.

When we need to compute a high power of a given number we may abridge the labour by a little management. We have seen that to multiply successively by several numbers gives the same result as to multiply at once by the continued product of those numbers ; that, for example, to multiply first by 3, then by 5, and then by 7, in succession, gives the same result as to multiply at once by 105 the product of $3 \times 5 \times 7$. Hence to multiply twice in succession by any number is the same as to multiply by the square of that number ; and to multiply by its fifth power is equivalent to five successive multiplications by the number.

Suppose then that we seek the thirteenth power of the number 3. Having obtained its square, 9, we multiply that square by itself to obtain 81, the fourth power of three ; multiplying 81 by itself we obtain 6561, the eighth power of 3 ; multiplying this again by 81, we get 531 441, the twelfth power of three, and this multiplied by 3 gives 1 594 323, the thirteenth power.

Examples.

Find the values of

5^{14}; 12^{16}; $(4\frac{1}{6})^{10}$; $(\frac{1}{3})^{7}$; 1.007^{19};
$.31^{11}$; $.023^{9}$; $.004^{18}$; 10.13^{17}.

In this way we come to recognise a truth which is, perhaps, the most important in the whole science of calculation, and to which the student will do well to give his most earnest attention: it is this, *That the product of two powers of the same number is another power of that number having for its index the sum of the indices of the two factors.* Thus the product of 7^5, viz. 16 807 by 7^3, that is by 343, gives 5 764 801, which is the eighth power of 7; that is to say

$$7^5 \times 7^3 = 7^8.$$

We must be careful to observe, that although the multiplication seem to be performed by addition, there is nothing of the nature of addition in the process. Many thoughtless persons imagine that the product of 7^5 by 7^3 should be 7^{15}; or that the sum of 7^5 and 7^3 should be 7^8; those who make such mistakes had better resume the study of the subject at the beginning.

Examples.

Perform the following multiplications symbolically:

$6^3 \times 6^4$ $(2\frac{1}{2})^9 \times (2\frac{1}{2})^{21}$ $73^3 \times 73^{11}$ $13^5 \times 13$
$17^4 \times 17^5 \times 17^7$ $5^5 \times 5^{11} \times 5^7$.

Since the product of 7^5 by 7^3 is 7^8, it follows that the quotient obtained on dividing the eighth power of 7 by the third power of 7, is the fifth power of the same number; or that

$$7^8 \div 7^3 = 7^5,$$

and similarly of other numbers and other powers; or in general that *The quotient obtained on dividing one power of a number by another power of the same number, is a power of that number, the index of which is the difference between the indices of the dividend and divisor.*

Examples.

Perform the following divisions symbolically :—

$$3^{17} \div 3^{5}\,; \qquad 7^{15} \div 7^{3}\,; \qquad (1\tfrac{3}{4})^{19} \div (1\tfrac{3}{4})^{17}\,;$$
$$13^{10} \div 13^{5}\,; \qquad 43^{20} \div 43^{4}\,; \qquad 9^{7} \div 19^{6}.$$

Sometimes we have to take the power of a number which is already the power of another number: thus we may have to compute the fifth power of 81, which itself is the fourth power of 3 ; and we may need to know what power of 3 the result may be. Now each multiplication by 81 is equivalent to four multiplications by 3, and therefore the fifth power of 81 must be the twentieth power of 3. Thus,

$$81^{5} = (3^{4})^{5} = 3^{4} \times 3^{4} \times 3^{4} \times 3^{4} \times 3^{4} = 3^{20}\,;$$

that is to say, *When a power of some number is raised to some other powers, the result is a power of the original number, having for its index the product of the two indices.*

Examples.

Exhibit symbolically the following complex powers:—

$$(2^{7})^{3}\,; \qquad (5^{6})^{4}\,; \qquad (17^{6})^{6}\,; \qquad (23^{5})^{7}\,;$$
$$(23^{7})^{5}\,; \qquad \{(13^{2})^{3}\}^{5}\,; \qquad \{(29^{7})^{5}\}^{3}.$$

When we have to multiply together powers of different numbers, no such facilities occur. Thus the 5th power of 3 multiplied by the 7th power of 11 cannot be obtained by adding the exponents or by multiplying the roots. But when the powers have the same index their product is that power of the product of the roots. Thus,

$$5^{7} \times 11^{7} = 55^{7},$$

and the product of 3^{5} by 11^{7} may be put in the form $33^{5} \times 11^{2}$, for 11^{7} may be regarded as the product of 11^{5} by 11^{2}, and $3^{5} \times 11^{7}$ as the product of the three factors 3^{5}, 11^{5}, and 11^{2} ; now the product of 3^{5} by 11^{5} is 33^{5}, so that

$$3^{5} \times 11^{7} = 3^{5} \times 11^{5} \times 11^{2} = 33^{5} \times 11^{2}.$$

Such transformations are often of great use in simplifying calculations.

EXAMPLES.

$3^7 \times 7^{11} \times 11^{13} =$;
$2^{10} \times 5^7 \times 3^8 =$;
$8^5 \times 5^{13} \times 13^2 =$;
$12^3 \times 4^2 \times 15^4 =$;

Sometimes, also, the division of one power of one number by some power of another number may be simplified. If, for instance, the 7th power of 12 be to be divided by the 3d power of 18, we may observe that 12 is the product of 2^2 by 3, while 18 is the product of 2 by 3^2; wherefore $12^7 \div 18^3 = 2^{14} \times 3^7 \div 2^3 \div 3^6 = 2^{11} \times 3$. Or again, if we have $20^5 \div 30^7$, we may put these under the forms $2^{10} \times 5^5$ and $2^7 \times 3^7 \times 5^7$, whence the quotient is $2^3 . 3^{-7} . 5^{-2}$, or $20^5 \div 30^7 = \dfrac{2^3}{3^7 \times 5^2}$.

EXAMPLES.

$35^4 \div 21^3 =$;
$360^2 \div 22^3 =$;
$18^4 \div 27^2 =$;
$135^5 \div 75^2 =$;
$175^4 \div 245^4 =$;
$105^6 \div 385^6 =$:
$1001^5 \div 1547^2 =$.

CHAPTER XX.

ON COMPOUND INTEREST.

D. If the interest on a sum of money have not been drawn when due, but have been left as an additional loan, also to receive interest, the resulting interest is said to be compound, because it is composed of the interest on the original capital and of the interest of that interest.

Thus, if three years ago a person had lent £10 000 to a bank at 3 per cent per annum, he was entitled, two years since, to have demanded £10 300 from the bank ; but if he left the whole money still in the banker's hands, he had a claim one year ago for £10 609 ; and to-day, if he have drawn nothing from the bank, he is entitled to this sum of £10 609, together with one year's interest thereon—that is, altogether, to £10 927‚27. He thus receives £927‚27 by way of interest for the use of his money during three years, whereas the mere interest of the principal sum of £10 000, or the simple interest, as it is called, amounts only to £900. The interest on the interest, then, has been £27‚27.

In all bargains for long periods of time compound interest is used ; thus, when a person, having a sum of money in hand for which he only expects to have use after several years, lends it to a company for that time certain, without intending to draw the interest annually, he naturally stipulates that each year's interest be added to the capital lent to receive interest also. The same principle is involved in transactions connected with life assurance ; for when we purchase a deferred annuity, an

assurance, or such like, the money which we pay is virtually laid out to accumulate at compound interest until the occurrence of a specified event.

In order to compute the amount of a given sum of money which has been lent for several years at compound interest, we may proceed in the manner just indicated; that is, we may compute the amount at each successive year.

Examples.

Required the amount of £7317, at 4 per cent per annum, in 7 years.

To what does £3207 amount in 6 years, at 3 per cent compound interest?

What is the amount of £78223 in 5 years, at 5 per cent?

But the calculation may be much more conveniently arranged. If interest be at 3 per cent, £100 must, after one year, amount to £103; and therefore any other sum of money lent at that rate of interest must be augmented in the ratio of 103 : 100; that is to say, the number which represents the sum of money must be multiplied by the fraction $\frac{103}{100}$, or, what is the same thing, by 1,03. This ratio 103 : 100 is called the annual rate of improvement of money.

Now, if interest continue at the same rate for another year, the product of this multiplication must be again multiplied by 1,03 to give the amount at the end of the second year. Thus in the second of the above examples, the amount at the end of one year must be £3207 × 1,03, and the amount at the end of the second year £3207 × 1,03 × 1,03; also the amount at the end of the third year must be—

$$3207 \times 1{,}03 \times 1{,}03 \times 1{,}03,$$

and so on. Now, to multiply three times in succession by 1,03 gives the same result as to multiply at once by the third

power of $1{,}03$, so that the amount at the end of 3 years may be expressed by

$$3207 \times 1{,}03^3,$$

and the amount at the end of 6 years by

$$3207 \times 1{,}03^6.$$

Hence it appears that in order to compute the amount of a sum of money in so many years, we must multiply the principal by that power of the rate of improvement which is indicated by the number of years.

For the computation of compound interest, then, it is convenient to have tables of the powers of the various rates of improvement. The construction of such tables is very simple; thus, to raise $1{,}04$ (the rate of improvement when interest is at 4 per cent), to its successive powers, we proceed as under :—

$$
\begin{array}{ll}
1{,}04^1 = & 1{,}04 \\
 & \phantom{1{,}0}416 \\
\hline
1{,}04^2 = & 1{,}0816 \\
 & \phantom{1{,}0}432\;64 \\
\hline
1{,}04^3 = & 1{,}1248\;64 \\
 & \phantom{1{,}0}449\;94\;56 \\
\hline
1{,}04^4 = & 1{,}1698\;58\;56 \\
 & \phantom{1{,}0}467\;94\;34\;24 \\
\hline
1{,}04^5 = & 1{,}2166\;52\;90\;24 \\
 & \phantom{1{,}0}486\;66\;11\;61 \\
\hline
1{,}04^6 = & 1{,}2653\;19\;01\;85 \\
 & \phantom{1{,}0}506\;12\;76\;07 \\
\hline
1{,}04^7 = & 1{,}3159\;31\;77\;92 \\
\text{\&c.} & \qquad \text{\&c.}
\end{array}
$$

Here the work has been carried to the tenth decimal place. So soon as we begin to cut off figures, the results become inaccurate, so that the last figures cannot be depended upon; hence it becomes necessary to carry our computations to several places beyond what we intend to record in our tables.

From this computation we find that £1 lent at 4 per cent compound interest for 7 years amounts to £$1{,}31593$, and we

therefore conclude that £7317 lent at the same rate must amount to

$$£7317 \times 1{,}31593 = £9628{,}673$$

in the same time.

Exercises.

Form tables of the amounts of £1 at the rates of 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, and 5 per cent for each year up to 20 years, carrying the results to 8 places of decimals.

By help of these tables, solve the following questions:—

What is the amount of £260 in 17 years at 3 per cent?

What is the amount of £5270 in 13 years at $3\frac{1}{2}$ per cent?

What will £1329 amount to 11 years hence at 4 per cent?

Required the amount of £2700 in 20 years at $4\frac{1}{2}$ per cent?

Required the amount of £97354,6 in 19 years at 5 per cent?

E. A more comprehensive view of the doctrine of compound interest may be obtained in this way:—

Suppose that a sum of money has been lent to some concern, with an agreement that it, with its accumulated interest, is to be repaid after a certain number of years; but that the interest is to fluctuate according to the actual market rate of interest in each year; and that, at the end of the time, we have to compute the amount.

In the first year we shall say that interest was at 3 per cent, in the second it had risen to 4, and had continued so during the third year, but that in the fourth year it had fallen to $3\frac{1}{2}$, and so on; and we shall take £1000 as the original sum. Then the amount, after one year, must have been £1000 × 1,03; after two years, £1000 × 1,03 × 1,04, etc.; so that, after four years, it was—

$$£1000 \times 1{,}03 \times 1{,}04 \times 1{,}04 \times 1{,}035.$$

Here we have the rates of improvement during the various years as continued multipliers. If the rates had been all alike—that is, if interest had continued at the same rate during the

whole time, the multiplier would have been the fourth power of the corresponding rate of improvement.

I have here to caution the student against a mistake which one is very apt to commit, viz. that of supposing that the *average* rate of interest during the four years may be substituted for the actual rates. He may readily satisfy himself that this is not the case by observing that the average rate of interest, in the above example, is $3\frac{5}{8}$, and that the fourth power of 1,03625 is not equal to the continued product $1{,}03 \times 1{,}04 \times 1{,}04 \times 1{,}035$.

The subject of compound interest is thus seen to be very intimately connected with the doctrine of powers.

Money lent at 5 per cent compound interest becomes almost doubled in 14 years, and more than doubled in 15 years, as is seen from the table of the powers of 1,05: it will then be again doubled in another 14 years, and so on; so that in about 42 years money accumulates to 8 times the original sum; in rather more than 56 years to 16 times; and in 71 years to 32 times the sum lent.

At 4 per cent the increase is, of course, less rapid. Nearly 18 years must elapse before the money be doubled; and at 3 per cent, $23\frac{1}{2}$ years are needed.

In some cases the interest is payable half-yearly or quarterly, and in the East even monthly; but this does not change the character of the calculations. Thus, in Constantinople interest is given at the enormous rate of $1\frac{1}{2}$ per cent per month; in order to find the amount at this rate in one year, we must raise 1,015 to the twelfth power.

CHAPTER XXI.

ON INVERSE POWERS.

D. The successive powers of a number or of a ratio form a series of terms in continued proportion, increasing if the ratio be greater than that of equality, and decreasing if it be less. We have had examples, in the preceding chapter, of series of quantities increasing in continued proportion; and we may easily find, in business, examples of decreasing series. I shall cite an illustration connected with pneumatics.

The air-pump is an instrument for extracting the air from any vessel. Air possesses the remarkable property of expanding to fill any space that may be left free to it, and no limit has, as yet, been found to this expansibility. In the construction of the air-pump advantage is taken of this quality. A cylinder, close at one end and fitted with a piston, is connected with the vessel from which the air has to be withdrawn: this vessel must be well closed, to prevent the ingress of the surrounding air. The piston having been pushed to the bottom of the cylinder, is drawn up, and this operation would leave the cylinder empty, were it not that the air contained in the vessel expands to fill the cylinder also. If, to take an example, the capacity of the vessel were nine pints, and that of the cylinder one pint, the air which at first filled nine pints has now expanded to fill ten, so that nine-tenths of the original quantity of air remain in the vessel. A stop-cock or valve at the bottom of the cylinder is now shut, to prevent the return of the air into the vessel, and the piston is thrust down to the bottom; an aperture being

opened to allow the escape of the air which was contained in the cylinder. This aperture having been shut, and the communication between the cylinder and the vessel having been again opened, the piston is once more drawn up. The air left in the vessel again expands to fill both the vessel and the cylinder, so that of whatever quantity of air which was in the vessel at the beginning of the stroke, only nine-tenths remain at the end of it; at the second stroke of the piston there will be left only nine-tenths of nine-tenths, that is, eighty-one-hundredths of the quantity of air originally in it. The quantities of air in the vessel at the ends of the successive strokes are thus:

$$\tfrac{9}{10} \qquad \tfrac{81}{100} \qquad \tfrac{729}{1000} \qquad \tfrac{6561}{10000} \qquad \tfrac{59049}{100000}, \quad \&c.$$

or, as we may write symbolically:

$$(\tfrac{9}{10})^1 \qquad (\tfrac{9}{10})^2 \qquad (\tfrac{9}{10})^3 \qquad (\tfrac{9}{10})^4 \qquad (\tfrac{9}{10})^5, \quad \&c.$$

This is an example of a series of decreasing continued proportionals.

When the ratio is expressed by an integer number, the powers go on increasing with great rapidity; when the ratio does not differ much from a ratio of equality, the progression is much more slow. In all cases, if we know the ratio and have one term of the progression, we can readily compute the succeeding term: nor is it more difficult, if one of the advanced terms be known, to compute the preceding one.

Thus, by repeatedly squaring, we find that the sixteenth power of 3 is 43 046 721, and we may thence obtain the fifteenth power by dividing this number by 3; from that again we can obtain the fourteenth, and so on: thus—

43 046 721	=	3^{16}	1 594 323	=	3^{13}
14 348 907	=	3^{15}	531 441	=	3^{12}
4 782 969	=	3^{14}	&c.		&c.;

and it is quite clear that, if we continue this division sufficiently, we shall return to 3 itself, and thence to unit: thus,

81	=	3^4	3	=	3^1
27	=	3^3	1	=	?
9	=	3^2			

From this example we see that 3 may, with propriety, be called the first power of 3; and we are led to propose the question, *What power of* 3 *is unit to be called?*

Now if we wish to define the fifth power of 3, we cannot say that it is the product obtained on multiplying 3 five times by itself, since there are only four multiplications; but we may define the fifth power of 3 to be the product obtained on multiplying *unit* five times by 3. Hence *unit* being unit not multiplied by 3, or, so to speak, *no* times multiplied by 3, may be called the *no* power, or, as we say, the zero power of 3, and thus the last terms of our progression may be written:

$$9 = 3^2 \qquad 3 = 3^1 \qquad 1 = 3^0.$$

The series of numbers 1, 3, 9, 27, &c., may be continued without limit by multiplication, and similarly the inverse series, 27, 9, 3, 1, may be continued indefinitely by division, the succeeding terms being $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, &c. These fractions, then, form a continuation, backwards, of the series of powers, and all together form one continuous progression, each member of which is three times the term on the one side, and the third part of the term on the other side,

$$\text{etc.} \quad 27, \quad 9, \quad 3, \quad 1, \quad \tfrac{1}{3}, \quad \tfrac{1}{9}, \quad \tfrac{1}{27}, \quad \text{etc.}$$

the double series being interminable either way.

The relation of the one branch of this series to the other exactly tallies with the relation of decimal fractions to decimal integers; the values of the figure 1, when written in the various places of the scale, being

$$\text{etc.} \quad 1000, \quad 100, \quad 10, \quad 1, \quad \tfrac{1}{10}, \quad \tfrac{1}{100}, \quad \tfrac{1}{1000}, \quad \text{etc.}$$

The fractions $\frac{1}{3}$, $\frac{1}{9}$, etc. form a continuation of the series of the powers of 3, and we give to them the name *inverse powers;* thus $\frac{1}{3}$ is called the *first inverse power of* 3, $\frac{1}{9}$ the *second inverse power,* and so on. Similarly ,0001 is the fourth inverse power of 10, and $\frac{1}{1024}$ the tenth inverse power of 2.

Inverse powers are denoted by writing a minus ($-$) before the index of the power; thus 3^{-4} stands for the fourth inverse power of 3, that is, for $\frac{1}{81}$, and may be taken to mean the result

obtained on dividing unit four times in succession by 3. Sometimes, for the sake of contrast, we write 3^{+4} for the fourth *direct* power of 3, as it may be called.

EXAMPLES.

Express in the common fractional form, and also in decimals, the values of the following inverse powers :

2^{-3}	3^{-8}	$(1\frac{2}{3})^{-4}$	$(\frac{2}{7})^{-4}$	$(4{,}57)^{-3}$
4^{-5}	9^{-4}	$(2\frac{3}{4})^{-1}$	$(\frac{1}{5})^{-7}$	$({,}37)^{-2}$
2^{-10}	10^{-5}	$(\frac{1}{3})^{-2}$	$(1{,}32)^{-2}$	$({,}017)^{-4}$

Among the examples above given, there are fractions, the inverse powers of which are required. It is most probable that the student has found no difficulty in dealing with them ; but lest any one should have been unable to satisfy himself on the subject, it may be well for me to make a few remarks on the inverse powers of ratios which are represented by fractions.

As an example, let us consider the ratio represented by the fraction $\frac{1}{3}$, that is, the ratio 1 : 3.

The second power of $\frac{1}{3}$ has been seen to be $\frac{1}{9}$, its third power $\frac{1}{27}$, and so on. Beginning, then, from say the fifth power, which is $\frac{1}{243}$, let us come backwards (or downwards, as we usually express it), through the series of powers, we have

$$\tfrac{1}{243} = (\tfrac{1}{3})^5 ; \qquad \tfrac{1}{81} = (\tfrac{1}{3})^4 ; \qquad \tfrac{1}{27} = (\tfrac{1}{3})^3 ;$$
$$\tfrac{1}{9} = (\tfrac{1}{3})^2 ; \qquad \tfrac{1}{3} = (\tfrac{1}{3})^1 ; \qquad 1 = (\tfrac{1}{3})^0 :$$

and here we cannot fail to observe, that since, in proceeding forwards, each power is the third part of that which preceded it, so, in coming backwards, each is three times the preceding ; the series then must go on in the same way, so that the continuation of it must be

$$\tfrac{1}{9} = (\tfrac{1}{3})^2 ; \qquad \tfrac{1}{3} = (\tfrac{1}{3})^1 ; \qquad 1 = (\tfrac{1}{3})^0 ;$$
$$3 = (\tfrac{1}{3})^{-1} ; \qquad 9 = (\tfrac{1}{3})^{-2} ; \qquad 27 = (\tfrac{1}{3})^{-3}.$$
$$\&c. \qquad \&c. \qquad \&c.$$

The inverse powers of the fraction $\frac{1}{3}$ are just the direct powers of the number 3 ; and contrariwise the inverse powers of the

number 3 are just the direct powers of the fraction $\frac{1}{3}$; and that this must be so, follows from the very obvious truth that if B be three times A, A must be the third part of B.

Similarly the direct powers of the fraction $\frac{3}{5}$ are the inverse powers of $\frac{5}{3}$, and for this simple reason, that if C be three-fifths of D, D must be five-thirds of C.

We have seen that the product of one power of a number by another power of the same number, is a third power of that number, having for its index the sum of the indices of the two factors; and thence we concluded, though perhaps rather hastily, that the quotient of one power by another power of the same number, is a power having for its index the excess of the index of the dividend above the index of the divisor; but it is clear that the argument whereby this is established can only hold good when there is an excess, that is, when the index of the dividend is greater than the index of the divisor. Let us now examine the case when the index of the dividend is not greater than that of the divisor.

If the index of the divisor be just equal to that of the dividend, as in this example, $5^3 \div 5^3$, it is clear that the quotient must be unit; at the same time it may be written 5^{3-3}, or 5^0; and here we have a confirmation of the propriety of calling unit the zero power of 5, or of any other number.

But if the divisor be a higher power than the dividend, as in this example $5^3 \div 5^7$, we cannot subtract the index 7 from the index 3. From 3 the utmost that we can subtract is 3 itself: having done that, we have still 4 subtractive, as it were, and therefore we write the quotient of 5^3 by 5^7, 5^{-4}. To divide by the 7th power of 5 gives the same result as to divide seven times in succession by 5; now, on dividing 5^3 seven times successively by 5 we obtain the quotients

$$25, \quad 5, \quad 1, \quad \tfrac{1}{5}, \quad \tfrac{1}{25}, \quad \tfrac{1}{125}, \quad \tfrac{1}{625};$$

or

$$5^2, \quad 5^1, \quad 5^0, \quad 5^{-1}, \quad 5^{-2}, \quad 5^{-3}, \quad 5^{-4}.$$

And here we see the propriety of indicating inverse powers by

prefixing the sign $-$ to the index. The same operation may be put thus:

$$\frac{5^3}{5^7} = \frac{5.5.5}{5.5.5.5.5.5.5} = \frac{1}{5.5.5.5} = 5^{-4}.$$

Hence, when one power of a number has to be divided by a higher power of the same number, we have to subtract the index of the dividend from that of the divisor in order to obtain the index of the inverse power which represents the quotient.

EXAMPLES.

Perform the following divisions:—

$3^4 \div 3^5$; $5^2 \div 5^4$; $13^{10} \div 13^{14}$; $28^7 \div 28^{13}$;
$41^3 \div 41^{19}$; $10^3 \div 10^{17}$; $(\frac{1}{2})^5 \div (\frac{1}{2})^7$; $(\frac{2}{3})^5 \div (\frac{2}{3})^6$.

E. Many writers on algebra, deceived by this and other analogous cases, have supposed that in the solution $5^3 \div 5^7 = 5^{-4}$ we really have subtracted 7 from 3, and found the remainder -4; and out of this misapprehension has arisen an extensive branch of analysis, the objects of which are *quantities less than nothing.*

The student may, however, easily detect the fallaciousness of such a statement as that we *can take the greater from the less,* by observing that the index 4 was in reality got by taking the 3 from the 7, and that the sign $-$ prefixed to it was obtained by considering the nature of the question.

It is convenient, for the sake of generalising our language, to call this operation the subtraction of 7 from 3, and to say that the remainder, on subtracting 7 from 3, is -4; just as it is convenient to speak of multiplying by one-half while the real operation is dividing by 2. A conventional misuse (or generalisation, as we call it) of language cannot change the nature of the things spoken of.

D. If we have to multiply a direct by an inverse power of the same number, we have only to regard the one as indicating

so many multiplications and the other so many divisions by that number, and to give the sign of the preponderance to the difference of the two indices. Thus $7^5 \times 7^{-3} = 7^{+2}$, and $9^{11} \times 9^{-17} = 9^{-6}$.

EXAMPLES.

$3^4 \times 3^{-2}$; $4^5 \times 4^{-1}$; $8^7 \times 8^{-5}$; $17^6 \times 17^{-9}$; $37^{11} \times 37^{-12}$; $23^{-6} \times 23^{15}$; $41^{-19} = 41^3$;

To preserve the same fictitious uniformity of language, we are said, in performing the operation $7^5 \times 7^{-3}$ to be adding -3 to 5, whereas in reality we take 3 from 5.

If we have to multiply one inverse power of a number by another inverse power of the same number, we must consider that both indicate division; so that $5^{-4} \times 5^{-3} = 5^{-7}$, for these factors may more properly be written

$$\frac{1}{5^4} \times \frac{1}{5^3} \text{ and the product } \frac{1}{5^7}.$$

EXAMPLES.

$7^{-2} \times 7^{-5}$; $(\frac{1}{2})^{-4} \times (\frac{1}{2})^{-9}$; $(\frac{3}{7})^{-3} \div (\frac{3}{7})^{-5}$.

But if we were to divide an inverse power by a direct power, the number of divisions is augmented; thus the division of 7^{-3} by 7^4 is in reality the division of the fraction $\frac{1}{7^3}$ by 7^4, and the quotient must necessarily be $\frac{1}{7^7}$, or 7^{-7}.

EXAMPLES.

$5^{-2} \div 5^2$; $7^{-5} \div 7^{+3}$; $13^{-7} \div 13^9$.

Lastly, when we have to divide by an inverse power, we multiply by the corresponding direct power: for the expression to divide by 5^{-4}, that is, by $\frac{1}{5^4}$, is only a conventional mode of saying to multiply by 5^4: hence we conclude that

$$5^3 \div 5^{-4} = 5^7.$$

In performing this operation we say that we subtract -4 from 3, whereas in reality we add 4 to 3.

The utility of, nay, even the necessity for, the use of these conventional modes of speaking will become apparent when we proceed to study universal arithmetic, commonly called algebra, in which numbers and ratios are represented by indefinite characters, so that we cannot know, until we come to apply ourselves to a particular case, which of two quantities may be the greater.

EXAMPLES.

$4^2 \div 4^{-1}$; $3^5 \div 3^{-3}$; $6^8 \div 6^{-8}$; $11^{-4} \div 11^{-7}$; $23^{-5} \div 23^{-2}$; $76^{-2} \div 76^{-23}$; $(\frac{2}{3})^5 \div (\frac{2}{3})^{-7}$.

When the power of any number is to be squared, cubed, or raised to any power, the index of the resulting power is the product of the two indices; and the truth of this is obvious when the powers are direct. Let us examine the case of inverse powers.

First let us take the example, "to find the second inverse power of the fifth power of three: that is $(3^5)^{-2}$."

To raise any number to its second inverse power is to divide unit twice by that number; wherefore

$$(3^5)^{-2} = 1 \div 3^5 \div 3^5 = 3^{-10}.$$

And thus we see that the inverse power of a direct power is an inverse power of which the index is the product of the two indices.

Next let us take the example, "to find the second power of the fifth inverse power of 3: that is $(3^{-5})^2$."

Here $(3^{-5})^2 = 3^{-5} \times 3^{-5} = 3^{-10}$, and we conclude, in general, that the direct power of an inverse power is an inverse power having its index the product of the two indices.

Lastly, let us take the example, "to find the second inverse power of the fifth inverse power of 3: that is $(3^{-5})^{-2}$."

Here $(3^{-5})^{-2} = 1 \div 3^{-5} \div 3^{-5} = 1 \times 3^5 \times 3^5 = 3^{10}$; whence

it appears that the inverse power of an inverse power is a direct power, having for its index the product of the two indices.

The four cases of composite powers deserve the student's close attention. They are exemplified thus:—

$$(3^{+5})^{+2} = 3^{+10}$$
$$(3^{+5})^{-2} = 3^{-10}$$
$$(3^{-5})^{+2} = 3^{-10}$$
$$(3^{-5})^{-2} = 3^{+10}$$

Examples.

$(4^3)^{-1}$; $(6^5)^{-3}$; $(9^{-7})^5$; $(11^4)^3$; $(13^{-6})^{-7}$; $(21^{-4})^{-9}$.

CHAPTER XXII.

ON THE COMPUTATION OF ANTICIPATED PAYMENTS (COMMONLY CALLED DISCOUNT).

D. WHEN a person who is entitled to receive a certain sum of money at some future time transfers or sells his right to another person, he does not receive the whole sum due, since no man would give one thousand pounds down for the right to draw one thousand pounds ten years hence. Of this we have had examples in the discounting of bills and promissory notes.

In the case of bills, the deduction is put as a discount off the sum mentioned in the bill, and the rate of *discount* is known; but we have often to compute the present value of a future payment, allowing interest at so much per cent; and it has already been shown that 5 per cent discount is a higher remuneration for the use of money than 5 per cent interest is.

Thus we may be asked, "*What is the present value of* £10 000 *due one year hence, when interest is at* 4 *per cent?*" In this case we are not to take 4 per cent discount off the £10 000, but we are to allow 4 per cent interest on the present value. The question, in fact, becomes this: "*What sum of money, laid out at* 4 *per cent, will amount to* £10 000?" Now £100 invested just now will amount to £104 one year hence: wherefore we have this proportion, as 104 is to 100, so is £10 000 to the sum which, laid out now at interest, will amount to £10 000 at the end of the year. That is to say, in order to obtain the present value we must divide £10 000 by the ratio 1,04, or, in other words, multiply £10 000 by the first inverse power of 1,04.

Similarly, if we seek the present value of a sum of money due two years hence, we must again divide by 1,04 ; that is to say, we must multiply the principal sum due by the second inverse power of 1,04.

And, in general, the present value of a sum of money, due several years hence, is to be obtained by multiplying by the proper inverse power of the rate of improvement of money. Thus the present value of £10 000, due 13 years hence, interest being supposed at 4 per cent, is represented by the expression,

$$£10\,000 \times 1{,}04^{-13}.$$

For the purpose of facilitating calculations of this kind, tables of the inverse powers of the various rates of improvement of money are prepared. The numbers given in these tables may also be called the present values of £1 due at a stated future time.

Thus for 4 per cent we make a table of the inverse powers of 1,04 : thus—

1,04	1,0000 0000 0000	= $1{,}04^{0}$
	46 6088 4660	
	615 4846 1548	
1,04	,9615 3846 1538	= $1{,}04^{-1}$
	57 7842 3118	
	245 5621 3 16	
	,9245 5521 3016	= $1{,}04^{-2}$

and so on.

Extensive tables of this kind are printed, but the student would do well to compute them for himself, as far, say, as to 20 years, just as he has done for the positive powers.

By help of such tables we can readily calculate the present value of any future payment: we have only to multiply the sum proposed by the proper inverse power of the rate of improvement of money.

Examples.

Required the present value of £6852 payable 2 years hence, interest being at 3 per cent.

Required the present value of £1562,4 due 7 years hence, interest at $3\frac{1}{2}$ per cent.

Required the present value of £3519,7 payable 13 years hence, interest being at 4 per cent.

What is the present value of £18 000 payable 17 years hence, interest being at $4\frac{1}{2}$ per cent?

What is the present value of £28 700 due 19 years hence, allowing interest at 5 per cent?

CHAPTER XXIII.

ON ROOTS AND FRACTIONAL POWERS.

D. When we know a number, it is very easy to find its square; we have only to multiply the number by itself; but it is not quite so easy to see how, knowing the square, we can discover the number of which that is the square.

If we wish to find that number of which 841 is the square, we may proceed to make trials, and after a few trials, we find that it is 29, since the square of 29 is just 841; 29 is called *the square root, the second root,* or often simply *the root of* 841.

Now although by guessing, and squaring, and guessing again, we may get to know the square root of any number which has one, this would be a very tedious and clumsy way of going to work, particularly if the number were large. Yet, after all, there is no other method known for obtaining a square root; we can only, by carefully arranging our trials, render the work rapid and certain.

One very obvious method of obtaining the square root of a number, is to resolve the number into its prime factors. Thus 441 is the continued product of $3 \times 3 \times 7 \times 7$; and these factors occur in pairs, so that the root must be the product of one of each pair of factors; in this case 3×7, or 21; and this plan may be used when we have a table of divisors; but from its very nature it can only be applied to numbers which have their factors in pairs: to such a number as 437, which is the product of 19 and 23, this method will not apply. Nay more, no number which has not its factors arranged in pairs, can possibly

have a square root; for we know that the products of no two sets of prime numbers can ever be alike.

The student may apply this method to the following

EXAMPLES.

Required the square roots of

36	169	529	2401	11025
64	225	900	9801	8281

Since the method of guessing and correcting our guess is the only practicable method of proceeding, our great business is to obtain some guide as to the first guess, and then some convenient way of estimating the corrections.

The decimal arrangement of numbers affords great facility for making the first guess. Since the square of 10 is 100, we readily see that, if a number have not more than two places of integers, its root can have only one place; as also, that since the square of 100 is 10 000, numbers of three and four places can be the squares only of numbers of two places; and in general, that for every two additional figures in the square, there must be one figure additional in the root.

If, then, we were required to find out the root of such a number as 27 447 121, we should count the number of its digits; if that number be even, the half of it is the number of integer places in the root; but if that number be odd, we add one to it before halving. Or more conveniently, we group the figures in pairs, beginning at the units, thus, 27 44 71 21, then for each group we must have one digit in the root.

In this example the root must have four places, that is, its highest figure must be in the place of thousands; or, in other words, the root must be between 1000 and 10 000. Now, on looking to 27, the highest group, we notice that it is somewhat above the square of 5 and considerably below the square of 6, and therefore we conclude that the root of which we are in search is between 5000 and 6000, being nearer the former; the highest figure of the root, then, is *certainly* 5.

Taking our first guess at 5000, let us see the extent of the error. On subtracting the square of 5000 from the given number, as in the margin, we find the defect to be 2 44 71 21, and the question becomes, "*What addition are we to make to the assumed root* 5000, *in order to make up this defect in the square?*" Now if we were to add 1000, that is, if we were to pass to 6000, the square would become 36 000 000, or 11 000 000 more than before; but we only want 2 447 121, and therefore we say, if the difference, 11 000 000, have been caused by an increase of 1000, what increase is needed to cause a difference of 2 447 121?

27 44 71 21
25 00 00 00
2 44 71 21

Although the increment of the square be not proportional to the increment of the root, we may, merely for the purpose of guessing, make the proportion, 11 000 000 : 2 447 121 : : 1000, the required correction; this we find to be about 200; and therefore we now try whether 5200 be nearly the root required.

On squaring this new number we find 27 040 000, so that the error is now reduced to 40 71 21. Now we have seen that the difference between the squares of two contiguous numbers is the sum of those numbers, and therefore the square of 53 exceeds the square of 52 by 105; that is, the square of 53·· must exceed the square of 52·· by 1 05· ···; so that again we may roughly make the proportion—

27 44 71 21
27 04 00 00
40 71 21

1 050 000 : 40 71 21 : : 100 : 38,

and so find that our next correction must be about 38; however, let us be contented with one additional figure, and try 5230.

On squaring and subtracting we find that the error is only 94 221. But a change of 10 in the root would cause a change of 104 700, that is $(5230 + 5240) \times 10$ in the square, and therefore we may make the proportion 104 770 : 94 221 : : 10 : 9, very nearly; and on trying 5329 we find that this is indeed the square root of the given number.

27 44 71 21
27 35 29 00
9 42 21

The student must be careful to observe that these proportions

are not *true*, they are only taken as rough guides in guessing the corrections to be applied to the assumed roots.

EXAMPLES.

Find in this way the roots of the following numbers :—

9 409	889 249	665 949 636
17 161	1 096 209	2 306 130 774 025
21 609	35 295 481	94 346 418 454 621 641
292 681	182 169 009	1 152 601 315 553 885 796

The chief labour in such calculations is that which attends the squaring of the roots successively assumed. This labour may be much reduced by taking advantage of the fact, that the difference between the squares of two numbers is the product of the sum by the difference of those numbers. Thus, to return to the former example, in order to obtain the square of 5200, we may add the product of 10 200 by 200 to the square of 5000; and so on, as shown in the margin. These operations are much less laborious than the actual squarings.

5•••²		=	25 •• •• ••
102••	× 2••	=	2 04 •• ••
52••²		=	27 04 •• ••
1043•	× 3•	=	31 29 ••
523•²		=	27 35 29 ••
10469	× 9	=	9 42 21
5239²		=	27 44 71 21

The steps of this calculation may be more compactly arranged thus : Having written the first assumed root 5000 twice, we multiply to obtain the square, 25 000 000, Adding the two 5000's together, and including the next correction 200, we obtain 10 200, which, multiplied by 200, gives 2 040 000 to be added to the square of 5000 in order to give that of 5300 ; and so on, as shown in the adjoining work. The formation of the successive terms may be more clearly shown by omitting the zeros altogether, or by

5•••	
5•••	
102••	25 •• •• ••
2••	2 04 •• ••
1043•	27 04 •• ••
3•	31 29 ••
10469	27 35 29 ••
9	9 42 21
	27 44 71 21

supplying their places with noktas, as in the margin, where it is seen that, while the root descends by one step at a time, the square descends two steps in the scale.

By this means we easily obtain the squares of the assumed roots: a very slight modification of the arrangement enables us to exhibit the successive errors in the squares.

5	27 44 71 21
5	25
102	2 44 71 21
2	2 04
1043	40 71 21
3	31 29
10469	9 42 21
9	9 42 21
10478	

Having pointed off the proposed number in groups of two figures each, we take the root of the square number immediately less than the number occupying the highest group, in this case 5, and write it to the left, and again below (omitting the zeros, though not forgetting their virtual presence). Multiplying we obtain the square, write it below the first group of the proposed number, and subtract in order to find the error of the square, in this example 2 447 121.

Adding the 5 to the 5 (truly 5000 to 5000), we obtain 10, which, when augmented by the next unknown correction, has to be multiplied by that correction, in order to give the correction of the square. It is only necessary, for the moment, to attend to the first part of the remainder, or *error*, as it may with propriety be called. Now it is our object to apply a correction to our assumed root sufficient to destroy or exhaust this error; the next figure must, then, be chosen so that when it is annexed to the double of that part of the root which has already been found, and the amount multiplied by it, the product should be equal to the error.

In this case we readily guess that the next figure is 2, since 10 goes rather oftener than twice in 24. Annexing the 2, and writing it also below, we multiply and obtain 204 (properly 2 040 000), the difference between the square of 50 and that of 52; and this subtracted from the preceding error, leaves now only the error 40 71 21.

In order to obtain the next correction, we add 102 and 2,

which make 104, just the double of 52, the assumed root ; we estimate how often this 104 goes in the error, and readily find the next figure 3 ; this we annex, and continue the operation, until the error be completely exhausted.

It is usual to keep a memorandum of the figures of the root as they are found : if such a memorandum be not kept, the root may be got by halving the final sum at the left hand.

EXAMPLES.

Extract the square roots of the following numbers :—

15 625	82 519 056
34 225	700 728 039 025
47 089	120 866 780 281
1 876 900	494 428 360 336
6 671 889	1 258 369 202 250 000
39 075 001	1 234 567 900 987 654 321
48 344 209	338 292 728 699 631 876

E. This method of obtaining the square root of a number is so concise and so rapid, that many writers consider it to be a *direct* process. Its truly tentative character may, however, be exhibited by such an example as the following :—

Let the root of 321 735 969 be required.

Here the highest group contains only one figure, viz. 3, so that the first figure of the root must be 1. Squaring this and subtracting, we find the remainder 221, and the divisor 2, whence we obtain the quotient 11. But we cannot have 11 in the next place ; the highest possible number that we can have there is 9, since anything more would convert the previous 1 into a 2. Let us then try 9 ; the product of 29 by 9 is greater than the error, so that 9 even is too much. Let us cancel the 9 and try 8 ; 8 times 28 are 224, which also is more than the

1	3 21 73 59 69	1
1	1	
2	2 21	

1	3 21 73 59 69	19
1	1	
29	2 21	
9	2 61	

error, so that 8 is too much; we must cancel it and try 7. Thus we see that, although with this arrangement the trials be very easily made, the process is virtually one of trial and error.

1	3 21 73 59 69	18
1	1	
28	2 21	
8	2 24	

It is quite clear that, since the utmost limit of the subsequent figures is 9999, etc., the divisor 2 in this example, with the addition of the next correction, can never come up to 3, and therefore the correction of the root can never be less than 221735969 ÷ 30000, while it can never be greater than 221735969 ÷ 20000. Thus we obtain two limits, 73911989 and 110867984, between which the correction must lie: but, from the very nature of the case, the next figure cannot be above 9, wherefore it must be one of the three 9, 8, 7.

Examples.

Required the square roots of

310993225; 738534976; 27950824225.

In the above class of examples the first remainder is large: let us now consider those cases in which the remainder is very small; as when we extract the root of 1370702529.

3	13 70 70 25 29	37023
3	9	
67	4 70	
7	4 69	
7402	1 70 25	
2	1 48 04	
74043	22 21 29	
3	22 21 29	

There, after the second subtraction, the remainder is only 1, and, on bringing down the next group, we have 170, which does not contain the divisor 74 with another figure annexed: hence

the next figure in the root must be zero; we therefore write 0, and bring down another group.

D. In all the preceding examples the proposed numbers are squares, and their roots have come out exactly in integers; we have now to consider those very numerous cases in which the roots of unsquare numbers are required.

When a board is made rectangular, we compute the number of square inches in its surface by multiplying the number of the linear inches in the length by that of the linear inches in the breadth. If the length and breadth be alike, the number of the square inches in the surface is the second power of the number of the linear inches in the side of the square: hence the problem in arithmetic, "*To extract the square root of a given number*," is almost identic with the geometrical problem, "*To construct a square which may contain a given quantity of surface.*"

If the question were proposed, "*what is the side of a square which contains 29 square inches?*" we should endeavour to obtain the answer by extracting the root of the number 29. Now the square of 5 is 25, and that of 6 is 36, wherefore the root of 29 must be intermediate between 5 and 6; that is to say, the side of the required square must be more than 5 but less than 6 inches; yet it cannot be represented by any fraction, since the square of a fraction which is in its lowest terms, cannot be simplified, and therefore never can be an integer number. Although, however, we cannot represent the side of this square accurately by a fraction, we may yet approximate as closely as may be required for any specific purpose.

The roots of unsquare numbers are called *Surds,* and are indicated by the sign $\surd$, which is a corruption of the initial letter *r* of the Latin word *radix,* a root. The root of 29 is written $\surd 29$.

In order to obtain an approximation to the root of an unsquare number we convert the remainder into hundredths of the unit, the next remainder into ten-thousandths, and so on, by

bringing down two zeroes at each step; and we continue this until we have attained to the required degree of precision; thus the process for the root of 29 stands as under:

5	29	5,3851648
5	25	
10,3	4,00	
3	3,09	
10,68	,9100	
8	,8544	
10,765	55600	
5	53825	
10,7701	177500	
1	107701	
10,77026	6979900	
6	6462156	
10,770324	51774400	
4	43081296	
10,7703288	869310400	
8	861626304	
10,7703296	7684096	

Examples.

Extract to six decimal places the square roots of the following numbers:—

10	93	257	458	955	95430
39	114	391	693	1472	954300
48	175	426	777	3001	1703152

E. When we extract a root in this way, to a considerable number of places, we find that, towards the end of the work, there has been a great deal of unnecessary labour: we may avoid this by a mode of shortening analogous to that which we employed for division in decimals.

Since, to return the root of 29, the last divisor, when augmented by the subsequent corrections, can never be so much as 10,7703297, and never less than 10,7703296, it follows

that the subsequent correction must be contained between the limits.

$$\frac{{,}00000007684096}{10{,}7703297} \text{ and } \frac{{,}00000007684096}{10{,}7703296}$$

or,

$$\frac{{,}7684096}{107703297} \text{ and } \frac{{,}7684096}{107703296}$$

Now the difference between two fractions having the same numerator, and of which the denominators differ by unit, is a fraction having the same numerator, and for its denominator the product of the two denominators; wherefore the error of either of these fractions cannot amount to so much as

$$\frac{{,}7684096}{107703297 \times 107703296}.$$

Thus it appears that we may safely use 107703294 as a common divisor for a considerable number of figures.

From this it is clear that when we have obtained one-half of the entire number of figures which we desire, we may cease to annex decimal places to the remainder, and may go on for the rest of the work as in shortened division. However, for the sake of avoiding the accumulation of errors in the last place, it is expedient to bring down one figure more before beginning to shorten.

The conclusion of the process for finding the root of 29 is as under :—

$$\begin{array}{r|r|l}
10{,}7703296 & 76840960 & 07134504 \\
\scriptstyle 4\ 0543170 & \underline{75392307} & \\
 & 1448653 & \\
 & \underline{1077033} & \\
 & 371620 & \\
 & \underline{323110} & \\
 & 48510 & \\
 & \underline{43081} & \\
 & 5429 & \\
 & \underline{5385} & \\
 & 44 & \\
 & \underline{43} &
\end{array}$$

whence $\sqrt{29} = 5{,}385164807134504.$

Examples.

Extract to eight decimal places the roots of

35	134	1594	7358	29930
78	356	2618	39501	128880

Expert multipliers can shorten the labour of extracting roots by observing, that after a few figures are found, it is quite safe to take as many additional figures, less one; and that therefore we are not tied to the slow process of going on one figure at a time. We may even begin with two groups, if we know the squares of the two earlier numbers: thus the extraction of the root of 29 may be done as follows:—

```
      5,3 | 29,00        |53,85164
      5,3 | 28,09
   ---------------------
   10,685  | 91 00 00
        85 | 90 82 25
   ---------------------------------
   10,7701 64 | 17 7500 0000
            1 64 | 17 6630 6896
   ---------------------------------
   10,7703 28       | 869 3104
```

And if we have an extensive table of squares, several groups may be taken at first, and thus the labour may be greatly lessened.

D. When the root of a fractional number is required, as of 8,37215, we proceed exactly in the same way, taking care to

```
   2     | 8,37215      | 2,893
   2     | 4            |
   ---------------
   4,8   | 4,37
     8   | 3,84
   ----------------
   5,69  |  5321
      9  |  5121
   -----------------
   5,783 |  20050
       3 |  17349, &c.
   -----------------
```

bring down two places at a time: this is obvious. But when the ratio is entirely fractional, we must consider more minutely the nature of the case, as in this matter beginners are very apt to fall into error.

Let it be proposed to extract the root of the decimal ,8.

If we make a first estimate, as learners often do, of ,2, because the square of 2 is 4 and that of 3 is 9, we go sadly wrong, for the square of two-tenths is not four-tenths, but four-hundredths; and thus we see that it is the root of ,80—that is, of eighty hundredths—that must be taken, and this is ,8, so that the operation is carried on thus:—

,8	,80	,894
,8	,64	
1,69	,1600	
9	,1521	
1,78 4	7900	
4	7136, &c.	

We must therefore point off a decimal fraction in groups of two figures each, beginning from the decimal point; and for this obvious reason that the square of tenths gives hundredths, the square of hundredths ten-thousandths, and so on. It is also to be observed, that the root of a fraction which is less than unit is greater than the fraction; thus the root of ,36 is ,6.

Examples.

Required the square roots of the following decimals:—

4,572	,81462	,21563	326,3158417	8,5473
3,6904	2,74613	1,00034	,000061538	,910768582
,1375	,021563	,99966	,587915203936	

When the root of a common fraction is wanted, we take the root of the numerator for a numerator, and the root of the denominator for a denominator; thus the root of four-ninths is two-thirds.

Examples.

Required the roots of the following fractions:—

$\frac{64}{49}$;	$\frac{36}{121}$;	$\frac{867}{1083}$;	,81;	,0529;
$\frac{1}{841}$;	$2\frac{1}{4}$;	$11\frac{1}{9}$;	$11\frac{14}{25}$;	$66\frac{15}{49}$.

But this method is only convenient when the numerator and denominator of the fraction reduced to its lowest terms, are both square numbers. In other cases we may convert the

fraction into an equivalent decimal, and then extract the root; or otherwise we may compute separately the root of the numerator and that of the denominator, and then divide; but this proceeding would be attended with great labour.

In general, the most convenient process is to convert the fraction into an equivalent one having its denominator a square number; and then having extracted the root of the numerator, to divide by that of the denominator. Thus if the root of the fraction *five-sevenths* were required, we may divide the root of 5 by the root of 7: thus

$$\sqrt{\frac{5}{7}}=\frac{\sqrt{5}}{\sqrt{7}},$$

or multiplying both numerator and denominator by 7, we may convert the fraction $\frac{5}{7}$ into $\frac{35}{49}$; whence

$$\sqrt{\frac{5}{7}}=\sqrt{\frac{35}{49}}=\frac{\sqrt{35}}{7}.$$

Examples.

Find the square roots of the following fractions :—

$\frac{1}{2}$; $\frac{5}{6}$; $\frac{17}{12}$; $\frac{3}{11}$; $1\frac{6}{7}$; $\frac{1}{15}$; $\frac{16}{19}$;
$1\frac{1}{20}$; $\frac{13}{25}$; $\frac{43}{45}$; $\frac{37}{54}$; $\frac{61}{72}$; $\frac{23}{192}$.

It may be remarked that, although on using the denominator of a fraction as the multiplier we must necessarily obtain a fraction having a square denominator, it is not always necessary to use so high a multiplier: thus in the example $\frac{17}{12}$, the multiplier 3 is quite sufficient. Whenever the denominator contains a square factor, it is only necessary to use the remaining factor: thus $12=2.2.3=4.3$; so, as 4 is already a square, we only multiply by 3.

E. It is sometimes desirable to express the root of an unsquare number by means of a common fraction; and although it be impossible to accomplish this with absolute precision, the attempt to do it leads to some useful knowledge.

Let us, for example, try to express the root of 11 by a com-

mon fraction. This root is more than 3, but less than 4; it is then 3 with some fraction. Let us try $3\frac{1}{3}$. The square of $3\frac{1}{3}$ is $11\frac{1}{9}$, so that $3\frac{1}{3}$ is rather much. Instead, however, of proceeding in this way, let us compute the root of 11 in decimals: it is 3·316624488, etc.; and let us then, by the process of Lord Brouncker (see Vol. I., p. 162), seek to convert this decimal into a common fraction. We find the quotients 3; 3, 6; 3, 6; 3, 6, etc.; and, what is very remarkable, the farther we continue the decimal the oftener is the group of quotients 3, 6, repeated. Indeed, it can be shown that this circulation must go on for ever.

By help of these quotients we can obtain the series of fractions which approach continually to the true value of $\sqrt{11}$, these fractions being alternately too small and too great; thus—

$$\begin{array}{ccccccccc} & & 3 & 3 & 6 & 3 & 6 & 3 & \\ \frac{0}{1} & \frac{1}{0} & \frac{3}{1} & \frac{10}{3} & \frac{63}{19} & \frac{199}{60} & \frac{1257}{379} & \frac{3970}{1197}, & \text{etc}; \end{array}$$

and if we were quite certain that the period of quotients recurs for ever, we could save ourselves a great deal of labour, because when once the period is discovered, we might continue it without farther calculation.

Yet, although it can be clearly shown that the quotients obtained in seeking the square root of an unsquare number, or of a fraction, do always recur in periods, and also that the last quotient of each period is invariably double of the integer part of the root, it would not do for us to presume, at present, on the truth of these laws. The knowledge, however, of the fact that such laws have been discovered may serve to stimulate us in the prosecution of our studies.

EXAMPLES.

Express, by the method of continued fractions, the values of the following surds:—

$\sqrt{5}$; $\sqrt{26}$; $\sqrt{31}$; $\sqrt{8}$; $\sqrt{82}$;

$\sqrt{143}$; $\sqrt{145}$; $\sqrt{\frac{3}{7}}$; $\sqrt{\frac{1}{2}}$; $\sqrt{\frac{17}{13}}$.

On squaring the series of fractions obtained by this process, it is found that the numerators of the errors in excess and in defect, recur again and again, while the denominators rapidly increase ; so that the actual errors decrease as we proceed along the series.

As we have often to compute and use the square roots of numbers, extensive tables of them have been printed ; of these Barlow's is perhaps the best.

CUBE ROOTS.

D. As with squares, so with cubes ; we have often to find a number from its cube. To compute the contents of an oblong block we multiply together the three numbers which express its length, its breadth, and its thickness ; the continued product is the number of cubic units contained in the block. When the three dimensions are all alike, the block is a cube, and its volume is represented by the third power of that number which represents its side. If, then, it were proposed to cut a cubic block of stone, to contain some number (say 2000) of cubic inches, it would be necessary to find that number of which the third power is the proposed 2000 : this number is called the *cube root of* 2000, and is denoted by the mark $\sqrt[3]{2000}$, read the *cube root*, or *the third root of* 2000.

The only process known for finding the cube root of a number is that of trial and error. We guess the root, cube the assumed number, and by considering whether this exceed or fall short of the proposed number, we judge of whether our estimate be too high or too low. The calculations attending this operation are necessarily longer, but the principles which regulate it are exactly the same as for the square root.

Since the cube of 10 is 1000, that of 100, 1 000 000, and so on, it follows that on grouping the figures of a number in threes from the units' place, we obtain a group for each figure of the

cube root. Hence, in order to find the cube root of such a number as

76009496256,

we part its digits in groups of three each, thus—

76 009 496 256,

the highest group, however, having one, two, or three figures, as the case may be, and then conclude that for each group there must be one figure in the cube root: the cube root of the above number, then, has four places of figures.

Now the number 76 in the highest group is more than the cube of 4 and less than the cube of 5, so that the cube root which we are seeking must be between 4000 and 5000.

On trial we find that the cube of 4000 is too small by 12 009 496 256. Now an addition of 1000 to the root brings the cube to be 125 000 000 000, and therefore causes a change of 61 000 000 000. Hence, as before, we may roughly state the proportion (leaving off 6 figures)—

$$\begin{array}{rr} & 76\ 009\ 496\ 256 \\ 4000^3 = & \underline{64\ 000\ 000\ 000} \\ \text{defect} = & 12\ 009\ 496\ 256 \end{array}$$

61 000 : 12 009 :: 1 000 : 200

to obtain an approximate correction. Adding this correction to the assumed root, we obtain 4200 for a new trial.

Cubing this number and subtracting we find the defect to be only 1 921 496 256. Now we have seen that to pass from the cube of one number to the cube of the next number we must add three times its square, three times itself and unit. To the cube of 42, then, we must add 5292, 126, and 1, in all 5419, in order to obtain the cube of 43; therefore an addition of 100 to the supposed root would cause an augmentation of 5 419 000 000 in the cube; but we need only 1 921 496 256; wherefore, in order

$$\begin{array}{rr} & 76\ 009\ 496\ 256 \\ 4200^3 = & \underline{74\ 088\ 000\ 000} \\ \text{defect} = & 1\ 921\ 496\ 256 \end{array}$$

to approximate to the next correction, we may state the proportion—

$$5\,419\,000\,000 : 1\,921\,496\,256 :: 100 : 30,$$

and may assume 4230 as the basis of a third trial.

Again cubing and subtracting we find the error to be reduced to 322 529 256. Now an addition of unit to 423 causes an augmentation of $3 \times 423^2 + 3 \times 423 + 1 = 538057$ to its cube, and therefore an addition of 10 to our assumed root would augment its cube by 538 057 000; but we only need an augmentation of 322 529 256; therefore from the proportion

$$\begin{array}{rr} & 76\,009\,496\,256 \\ 4230^3 = & 75\,686\,967\,000 \\ \hline \text{defect} = & 322\,529\,256 \end{array}$$

$$538\,057\,000 : 322\,529\,256 :: 10 : 6$$

we obtain, as the next correction, roughly, 6, and on trial we find that the cube of 4236 is exactly the proposed number.

Examples.

Compute in this way the cube roots of the following numbers:—

35937	1367631	20570824	640503928
68921	3048625	198155287	741217625

The principle which guides us in this operation is simple: we make a trial, ascertain the error in the cube, thence estimate a correction, and try again; but the labour is great, and becomes excessive when the figures in the root are numerous. In order to lessen this labour we endeavour to form the cube of the newly assumed root from the cube of the preceding one. This we can accomplish by attending to the composition of the cube of the sum of two numbers.

The business is, having the cube of a number, such as 423, to pass to the cube of another number formed by annexing a lower figure, such as 4236; in reality, to pass from the cube of

4230 to that of 4236; as well as to prepare for passing onwards if need be.

Beginning at the first step, let us form the cube of 42 from that of 4, or rather of 40. Using the Arab nokta for the sake of clearness, we have

$$\begin{array}{rcr}
4{\cdot}^3 & = & 64{\cdot}{\cdot}{\cdot} \\
3\times 4{\cdot}^2\times 2 & = & 96{\cdot}{\cdot} \\
3\times 4{\cdot}\;\;\times 2^2 & = & 48{\cdot} \\
2^3 & = & 8 \\
42^3 & = & 74088
\end{array}$$

If we wish to proceed onwards we must have the square of 42, so that it is convenient to form it also as we go along. The work then would stand—

$$\begin{array}{rcrcrcr}
4{\cdot}^2 & = & 16{\cdot}{\cdot} & \qquad & 4{\cdot}^3 & = & 64{\cdot}{\cdot}{\cdot} \\
2\times 4{\cdot}\;\;\times 2 & = & 16{\cdot} & & 3\times 4{\cdot}^2\times 2 & = & 96{\cdot}{\cdot} \\
2^2 & = & 4 & & 3\times 4{\cdot}\;\;\times 2^2 & = & 48{\cdot} \\
 & & & & & & \\
42{\cdot}^2 & = & 1764{\cdot}{\cdot} & & 42{\cdot}^3 & = & 74088{\cdot}{\cdot}{\cdot} \\
2\times 42{\cdot}\;\;\times 3 & = & 252{\cdot} & & 3\times 42{\cdot}^2\times 3 & = & 15876{\cdot}{\cdot} \\
3^2 & = & 9 & & 3\times 42{\cdot}\;\;\times 3^2 & = & 1134{\cdot} \\
 & & & & 3^3 & = & 27 \\
423{\cdot}^2 & = & 178929{\cdot}{\cdot} & & 423{\cdot}^3 & = & 75686967{\cdot}{\cdot}{\cdot} \\
2\times 423{\cdot}\;\;\times 6 & = & 5076{\cdot} & & 3\times 423{\cdot}^2\times 6 & = & 3220722{\cdot}{\cdot} \\
6^2 & = & 36 & & 3\times 423{\cdot}\;\;\times 6^2 & = & 45684{\cdot} \\
 & & & & 6^3 & = & 216 \\
4236^2 & = & 17943696 & & 4236^3 & = & 76009496256
\end{array}$$

If, instead of computing the square of the assumed root, we compute three times that square, so as to have it ready for multiplication by the next correction, the work takes a more symmetrical form, and may be arranged concisely as under.

Having prepared five columns, the first to receive the number 6, the second to receive 6 times the root, the third to receive 3 times the square of the root, the fourth to receive the cube of the root, and the fifth a memorandum of the root itself;

we write 6 times the first assumed figure of the root, in this case 24, at the top of the second column; 3 times the square, in this case 48, at the top of the third column; the cube of it, 64, at the top of the fourth column; and make a memorandum of the root itself at the top of the fifth column: we also write the number 6, the product of $1 \times 2 \times 3$, at the top of the first column, in order to complete the line.

6	6 Root.	3 Root ^{2}.	Root ^{3}.	Root.
6	24 1 2	48 4 8 12	64 9 6 48 8	4 2
6	25 2 18	52 92 756 2 7	74 088 1 587 6 11 34 27	4 2 3
6	25 38 36	53 678 7 152 28 108	75 686 967 322 072 2 456 84 216	4 23 6
6	25 416	53 831 088	76 009 496 256	4 236

Having now determined on the next figure of the root, or the *correction*, as we may call it, we write it below the root, and one step out to the right; then, multiplying each number in the first line, excepting that in the fourth column, by this correction, we write each product in the column towards the right hand: in this way we obtain the second line of numbers. We next multiply the *half* of each number in the second line, with the same exception as before, by the correction, placing the product again in the next columns: lastly, we multiply the

third part of the number found in the third line by the correction, and place the product in the fourth column, as shown in the work; then summing up the numbers found in each column we obtain the number 6 unaltered, 6 times the new root, 3 times the square of the new root, the cube of the new root, and in the fifth column the new root itself, so that we are now ready to proceed with the second correction. In the adjoining work the zeroes are omitted for the sake of clearness: when the correction has one additional figure, as in the example given, the numbers proceed one place out at each step; but when the correction has two additional figures, the results go two steps out, and so on.

Before proceeding to study the application of this process to the extraction of cube roots, the student may exercise himself in computing by its help the cubes of a few numbers.

EXAMPLES.

Compute the cubes of the following numbers:—

1476; 3905; 6143; 76093; 207583.

Let it now be proposed, as before, to extract the cube root of the number 76009496256.

We write the given number at the top of the fourth column, grouping its digits in threes for the sake of convenience, and, having obtained the first figure of the root, we subtract its cube from the proposed number so as to obtain the error, and this error takes the place occupied by the cube of the first figure in the preceding operations. Our business now is to apply such corrections as may exhaust this error 12009496256. Now it is clear that the greater part of the augmentation of the cube is the product of thrice the square of the already assumed root by the correction; wherefore the correction may be roughly estimated by dividing the error by the number in the third column; in our example by dividing 120, &c., by 48. We thus obtain the next figure, 2.

6	6 Root.	3 Root².	Error.	Root.
			76 009 496 256 64	
6	24 1 2	48 4 8 12	12 009 9 6 48 8	4 2
6	25 2 18	52 92 756 2 7	1 921 496 1 587 6 11 34 27	42 3
6	25 38 36	53 678 7 152 28 108	322 529 256 322 072 2 456 84 216	423 6
6	25 416	53 831 088	0	4236

Proceeding as just explained, we find the augmentation of the cube, and therefore the deduction from the error, to be $9600 + 480 + 48$; and this *subtracted* from the previous error, reduces the error in the cube to 1921, &c. Dividing now 19214, &c. by 5292, we get the next figure 3, and with it we proceed just as before, continuing the operation until the error be exhausted. In the actual work it is convenient to omit those zeroes which are only needed to indicate the rank of the products; and it is also better to bring down only one group at a time than to write out the whole error.

EXAMPLES.

Find the cube roots of the following numbers :—

13 824	597 619	10 581 347 558 664
39 304	3 506 295	440 728 454 858 301
103 823	2 232 681 446	153,6921057
238 328	630 084 028 517	513 518 298 333 039

If there be a remainder, that is to say, if the proposed number be not a cube, we must annex decimal places, and continue the work until the required degree of precision be obtained.

E. When we have to extract a cube root true to a prescribed number of places, we may shorten the work, by attending to the limits within which the correction must necessarily lie.

In order fully to see the character of this shortening, let us take as an example the extraction of the cube root of 17.

6	6 Root.	3 Root².	Error.	Root.
			17, 8,	
6	12, 3,0	12, 6,0 ,75	9,000 6,0 1,50 ,125	2, ,5
6	15,0 ,42	18,75 1,050 147	1,375 000 1,312 5 36 75 343	2,5 ,07
6	15,42 72	19,814 7 18 504 4	,025 407 23 778 11	2,57 12
6	15,427 2 486	19,833 208 1 250	,001 618 1 607	2,571 2 81
6	15,427 686	19,834 458	,000 011	2,571 281

On trying the number 2 as a root, we find the error 9, and the divisor 12, from which we obtain the approximate correction ,7; this correction, however, turns out to be too great; we then cancel the work depending on this correction, and try ,6; but

this is also too great, so we try the correction ,5, and find the results as given in the adjoining working. The error is now 1,375, and the divisor 18,75, whence we obtain the approximate correction ,07333, &c.; but it is quite obvious that we cannot use the whole of this correction, since the multiplication of 18,75 by it would alone exhaust the whole error, and the parts of the augmentation brought up from the 15,0 and the 6 would be in excess. The correction obtained by dividing the error by three times the square of the assumed root must always be too great; of this we have already had an example. But it is not convenient to cancel our work, particularly when it has become heavy, and we therefore look for some criterion, by which we may judge whether the first figure 7 of the above correction be or be not too great. Now we see that the correction can not exceed ,09999, &c., that is, that its utmost limit is ,1. In that extreme case the numbers in the third column would stand as in the margin, so that the correction would be multiplied by the sum of 18,75, the half of 1,50, and the third part of 3, that is, by 19,51, and this being the greatest possible multiplier of the correction, it follows that the quotient obtained on using it as a divisor, is the least possible value of the correction; wherefore the correction which we are seeking must lie between

$$\begin{array}{r} 18,75 \\ 1,50 \\ 3 \end{array}$$

$$\begin{array}{r} 18,75 \\ ,75 \\ 1 \\ \hline 19,51 \end{array}$$

$$\frac{1,375}{18,75} = ,07333 \text{ and } \frac{1,375}{19,51} = ,07047.$$

so that the next figure of the root is *certainly* 7.

On applying this correction, we find the error ,025 407, and the *least* divisor 19,8147, while the greatest divisor is 19,8147, together with half of 15,42 multiplied by ,01 and the square of ,01, in all, 19,8919; wherefore the next correction must be between the limits

$$\begin{array}{r} 19,8147 \\ 771 \\ 1 \\ \hline 19,8919 \end{array}$$

$$\frac{,025407}{19,8147} = ,001282, \text{ and } \frac{,025407}{19,8919} = ,001277;$$

so that we are now *certain* that the two next figures of the root are 12. We have thus obtained the root true to four decimal places, and it is apparent that although the two

last figures of the error had been rejected, this correction would not have been affected. Hence it seems that, in this example, we need not carry the error out to more decimal places than we wish in the root. Supposing that we limit ourselves to six places of decimals, we need not continue the error to more places than we have already.

The process after this is quite clear ; the only thing to be observed is, that, as soon as we begin to leave off the decimals, the products shorten very rapidly ; at the last operation in our example only the first line of products tells on the sixth decimal place.

If we divide the number found at the bottom of the third column by 3, we obtain the square of the cube root, in this example 6,611 486.

D. When we seek the cube root of a decimal fraction without integers, we must be careful to group its digits in threes from the decimal point, because the cube of two *tenths* is eight *thousandths*, and so on.

EXAMPLES.

Compute to ten decimal places the cube roots and the squares of the cube roots of the following numbers :

2 ; 3 ; 10 ; 1375 ; 73,6184692 ;
484,024390787221 ; ,7854 ; ,37 ; ,0329651 ;
,434294481903 ; ,00736849 ; 1,0035 ;
,0004962584398.

To compute the cube root of a common fraction, we may proceed, as in the case of square roots, to extract the cube roots of the numerator and of the denominator separately, and then divide the one root by the other ; thus :

$$\sqrt[3]{\frac{5}{7}} = \frac{\sqrt[3]{5}}{\sqrt[3]{7}}.$$

But this process is laborious, except for those fractions of which

the denominators are cubes. However, every fraction can be converted into an equivalent fraction, having its denominator a complete cube. For example, by taking 49 times the numerator, and 49 times the denominator, the fraction $\frac{5}{7}$ becomes $\frac{245}{343}$, whence

$$\sqrt[3]{\frac{5}{7}} = \sqrt[3]{\frac{245}{343}} = \frac{\sqrt[3]{245}}{7};$$

and thus we have only to compute the cube root of 245, and divide that cube root by 7.

If the denominator be a prime number, or if it be a composite number having none of its factors repeated, the only expedient is to multiply by the square of that denominator. If, for example, $\sqrt[3]{\frac{13}{15}}$ were proposed, we must use the multiplier 15^2 or 225. But if any of the factors of the denominator be repeated, we may use a smaller multiplier: thus for the fraction $\frac{7}{12}$ we may use the multiplier 18; because 12 being the continued product 2.2.3, needs only the additional factors 2.3.3 to have three factors of each kind.

Examples.

Prepare the following fractions for the extraction of their cube roots:

$\frac{15}{32}$	$\frac{233}{300}$	$\frac{5}{9}$	$\frac{431}{484}$
$\frac{43}{54}$	$\frac{311}{847}$	$\frac{19}{50}$	$\frac{503}{507}$
$\frac{103}{147}$	$\frac{3763}{4000}$	$\frac{73}{75}$	$\frac{4373}{3971}$

On converting the cube root of an un-cube number into common fractions by the method of Brouncker, we obtain a series of quotients, but these never recur in groups, and thus we have not that facility for extending the precision of our operations which we had in the case of square roots.

FOURTH ROOTS.

The fourth root of a number may be found by extracting its square root, and then the square root of that again. It may also be found directly by an extension of the process which we have used for cube roots. Having separated the figures into groups of four each, counting either way from the decimal point, we seek that number, of which the fourth power is immediately less than the number represented by the figures in the highest group, and this is the highest or first figure of the root.

In order to be in a position to estimate what the second figure should be, we must recur to the formation of the fourth power of the sum of two numbers. That fourth power is made up of the fourth power of the first number, 4 times the cube of the first number multiplied by the second, 6 times the square of the first multiplied by the square of the second, 4 times the first multiplied by the cube of the second, and lastly, the fourth power of the second number.

Suppose that the first figure of the root is 7, and that the second figure is estimated at 3, we have to pass from the biquadrate of 7, viz. 2401, to the biquadrate of 73 (properly from 70^4 to 73^4). The first part of the difference is the product of *four times the cube of the assumed root* 7 *by the correction* 3 ; wherefore if we have to continue our work over many trials, we must provide the means of obtaining and recording *four times the cube of the root* as that root is found. Now the process which we have already followed enables us to compute the cube of the root, and therefore we have only to quadruple all the numbers in the first four columns of the scheme.

In preparing, then, to extract the fourth root, we arrange six columns, the first to contain the number 24, the second to contain 24 times the root, the third for 12 times the square of the root, the fourth for 4 times its third power, the fifth for the fourth power, and the sixth for a memorandum of the root itself.

Each number, in the first line, has to be multiplied by the correction, the product being removed into the adjoining column to the right ; these products form the second line. The half of each number in the second line has to be multiplied by the correction, the product being again advanced into the adjoining column ; the third part of each number in the third line has to be treated in the same way ; and, lastly, the fourth part of the solitary number in the fourth line, as shown in the adjoining work, in which, for the sake of clearness, the zeroes are omitted.

24	24 Root.	12 Root^2.	4 Root^3.	Root^4.	Root.
24	168 72	58 8 5 04 108	1 372 176 4 7 56 108	24 01 4 116 264 6 7 56 81	7 3
24	1752	63 948	1 556 068	28 398 241	73

The mode of applying this arrangement to the extraction of fourth roots is quite analogous to that for the extraction of cube roots : so much so that it is almost unnecessary to say a single word in explanation. The subjoined example of the extraction of the fourth root of 59 may suffice.

Here the first figure of the root being 2, we have the error 43, and the least value of the divisor 32, wherefore the correction must be less than $\frac{43}{32}$, or 1,34, etc.

E. By the very same reasoning as before we can show that the greatest possible multiplier of the correction is made up of the number in the 4th column, augmented by one-half of that in the 3d, one-sixth of that in the 2d, and one twenty-fourth of that in the 1st, multiplied respectively by the first, second,

and third powers of unit in the place last found ; this multiplier in the present instance is

$$,32 + \frac{1}{2},48 + \frac{1}{6},48 + \frac{1}{24},24 = ,65,$$

wherefore the least possible value of the correction is

$$\frac{43}{65}, \text{ or } ,69, \text{ etc.}$$

The correction thus lies between the limits ,69 and 1,34 ; however, we know that the correction is not so much as 1, and

24	24 Root.	12 Root².	4 Root³.	Error.	Root.
				59 16	
24	48, 16,8	48, 33,6 5,88	32, 33,6 11,76 1,372	43, 22,4 11,76 2,744 ,2401	2, ,7
24	64,8 1,68	87,48 4,536 588	78,732 6,1236 ,15876 137	5,8559 5,51124 ,21433 370 3	2,7 ,07
24	66,48 336	92,0748 ,09307 3	85,01573 ,12890 7	,12660 ,11902 9	2,77 ,0014
24	66,5136 211	92,16790 585	85,14470 811	,00749 ,00749	2,7714 ,000088
24	66,51571	92,17375	85,15281	,00000	2,771488
		7,68114	21,28820		

therefore that the next figure of the root must be one of the four 6; 7, 8, 9.

On trying 7 we find the error to be 5,8559. The least multiplier is 78,732, and the greatest is found thus :

$$\begin{array}{llll} & 78{,}732 & & = 78{,}732 \\ \frac{1}{2} & 87{,}48 & \times\ {,}1 & = 4{,}374 \\ \frac{1}{6} & 64{,}8 & \times\ {,}1^2 & = 108 \\ \frac{1}{24} & 24 & \times\ {,}1^3 & = 1 \\ \hline & & & \ 83{,}215 \end{array}$$

wherefore the correction lies between the two limits

$$\frac{5{,}8559}{78{,}732} = {,}074, \text{ \&c.; and}$$

$$\frac{5{,}8559}{83{,}215} = {,}070, \text{ \&c.}$$

We therefore conclude that the next figure of the root must be 7, and proceed accordingly.

It is only at the beginning that we have to attend to these limits ; they rapidly approach each other as the work proceeds. When the required degree of precision has been obtained, we can find the second and third powers of the fourth root by dividing the numbers at the bottoms of the third and fourth columns by 12 and by 4 respectively.

D. EXAMPLES.

Required the fourth roots of the following numbers, and also the second and third powers of those roots.

76979	286,573	8,6938	,729637	,0187251
373	102,437	2,69305	,1827364	,0032749

FIFTH ROOTS.

From these examples it is apparent that the processes for extracting the second, third, and fourth roots are merely cases of one general process applicable to all roots. This process I first published in a small treatise *On the Solution of Equations of all Orders*, *Edinburgh* 1829, but then only incidentally, and without the full details which are now given. It may be enough for me to proceed one step farther, and exemplify the extraction of the fifth root.

When an addition is made to the root, its fifth power receives an augmentation, of which the first part is the product of five times the fourth power of the root by the increment of the root; wherefore, in preparing a scheme for computing and recording the fifth powers of the successive values of the root, we must compute and record the values of five times their fourth powers; so that the numbers in the previous scheme for fourth powers must be multiplied by 5, and the titles of the columns must stand

120 120 *root* 60 *root*^{2} 20 *root*^{3} 5 *root*^{4} *root*^{5}.

These titles may be obtained most readily from the highest power backwards: thus we have 5, $5 \times 4 = 20$; $20 \times 3 = 60$; $60 \times 2 = 120$; $120 \times 1 = 120$.

After what has already been said, even an example is hardly needed to render the process clear.

Let it be required to extract the 5th root of 2,71828.

120.	120 Root.	60 Root^2.	20 Root^3.	5 Root^4.	Error.	Root.
					2,71828 1	
120	120 24,0	60 24,0 2,40	20 12,0 2,40 ,160	5, 4,0 1,20 ,160 80	1,71828 1,0 ,40 ,080 80 32	1, ,2
120	144,0 2,40	86,40 2,880 240	34,560 1,7280 2880 16	10,3680 ,69120 1728 19	,22996 ,20736 691 12	1,2 ,02
120	146,40 ,1680	89,3040 ,20496 12	36,31696 ,12503 14	11,07667 5084 9	,01557 ,01551 4	1,22 ,0014
120	146,5680 24	89,50908 29	36,44213 18	11,12760 7	,00002 2	1,2214 ,000002
120	146,56824	89,50937	36,44231	11,12767	,00000	1,221402
		1,49182	1,82212	2,22553		

The law of the formation of these schemes is so clearly seen from the last example that the student can experience no difficulty in applying it to still higher roots, and in obtaining, not only the root, but all its inferior powers.

EXAMPLES.

	2,841	$\sqrt[7]{,002976}$
$\sqrt[7]{2193}$	$\sqrt[11]{3,02573}$	$\sqrt[8]{,091764}$
$\sqrt[6]{876,17}$	$\sqrt[10]{1,68497}$	$\sqrt[13]{1,025736}$

FRACTIONAL POWERS.

We have seen that when a power of any number is itself to be raised to a power, the result is a power of the original number, of which the index is the product of the two indices; for example, that the third power of the fifth power of 7 is the fifteenth power of 7, or

$$(7^5)^3 = 7^{15}.$$

It is then clear that the third root of the fifteenth power of 7 is the fifth power of 7; or

$$\sqrt[3]{(7^{15}} = 7^5);$$

and we might be tempted to state the general law that the root of any power of a number is a power of that number, of which the index is the quotient of the index of the power by the index of the root. This law holds true of the above and of many other examples; but when the index of the power is not divisible by the index of the root, there is some difficulty in seeing how the law is to be applied.

If, for example, we have to extract the cube root of the sixteenth power of 7, we should, according to this law, obtain that power of 7 of which the index is $5\frac{1}{3}$.

$$\sqrt[3]{(7^{16})} = 7^{\frac{16}{3}} = 7^{5\frac{1}{3}}.$$

Since we have had no instance of such *fractional indices*, we can

only endeavour to interpret their meaning by studying the nature of the operation which has given rise to them.

The symbol $7^{\frac{16}{3}}$ is but another way of writing the third root of the sixteenth power of 7; it is simply equivalent to $\sqrt[3]{}$

In the same way $13^{\frac{2}{5}}$ would mean the fifth root of the second power of 13, or $\sqrt[5]{(13^2)}$; so also the fourth root of 11, or $\sqrt[4]{11}$, may be written $11^{\frac{1}{4}}$.

Viewed in this light, these fractional indices present to our minds only another notation of roots, more convenient in form than the notation by help of the contracted *r;* but when examined more minutely, they unfold some of the most recondite yet most practically useful properties of numbers.

Since the fourth power of 7 is 2401, the fourth root of 2401 is 7, or, as we write it,

$$7 = 2401^{\frac{1}{4}}.$$

Four successive multiplications by 7 produce the same effect as one multiplication by 2401 ; twelve multiplications by 7 the same effect as three multiplications by 2401 ; and so on : so that the multiplier 2401 may be said to be four times as powerful as the multiplier 7, or the multiplier 7 may be said to have only one-fourth part of the power of the multiplier 2401, and hence the propriety of saying that 7 is the one-fourth power of 2401.

Again, the third power of 7 is 343 ; and on comparing the effect of multiplications by 343 and by 2401, we observe that four successive multiplications by 343 produce the same effect as three multiplications by 2401 ; so that the multiplier 343 may be said to have only three-quarters of the power of the multiplier 2401, or

$$343 = 2401^{\frac{3}{4}};$$

or otherwise, that the multiplier 2401 has four-thirds of the power of the multiplier 343 ; that is

$$2401 = 343^{\frac{4}{3}}.$$

From this consideration it follows that the third root of the fourth power of a number is the same with the fourth power of the third root of that number, so that we may write indifferently,

$$2401^{\frac{3}{4}}; \quad \sqrt[4]{(2401^3)}; \quad (\sqrt[4]{2401})^3.$$

It also follows that the third root of the fourth power is the same with the sixth root of the eighth power, for eight multiplications by 343 produce the same effect as six multiplications by 2401. Hence we may put, instead of the fractional index $\frac{3}{4}$, any other fraction which is equivalent to it; thus,

$$\sqrt[4]{(2401^3)} = 2401^{\frac{3}{4}} = 2401^{\frac{300}{400}} = \sqrt[400]{(2401^{300})}.$$

From this we obtain the method of multiplying together two fractional powers of the same number. If we be required to multiply $5^{\frac{3}{7}}$ by $5^{\frac{2}{3}}$, we have only to change the indices into equivalent fractions having a common denominator; in this way the factors become $5^{\frac{9}{21}}$ and $5^{\frac{14}{21}}$; the one is thus the ninth and the other the fourteenth power of the twenty-first root of 5, so that their product must be the twenty-third power of the same root; that is,

$$5^{\frac{3}{7}} \times 5^{\frac{2}{3}} = 5^{\frac{9}{21}} \times 5^{\frac{14}{21}} = 5^{\frac{23}{21}}.$$

We also see that when the numerator of the index is greater than the denominator, the power may be regarded as the product of an integer by a fractional power; thus,

$$5^{\frac{23}{21}} = 5^{1\frac{2}{21}} = 5^1 \times 5^{\frac{2}{21}}.$$

The process which I have explained for finding the root of a number gives at the same time all the inferior powers of the root, so that by its help we are able to compute directly the values of all fractional powers of which the index is less than unit: by attending to the above remark we can compute the values of all fractional powers whatever.

EXAMPLES.

$$6^{\frac{2}{3}};\quad 4^{\frac{5}{7}};\quad 12^{\frac{5}{3}};\quad 7^{\frac{2}{9}};\quad 26^{\frac{4}{7}};\quad 22^{\frac{6}{5}};$$

$$7214^{\frac{6}{7}};\quad 3{,}592^{\frac{11}{9}};\quad 1{,}6305^{\frac{13}{7}};\quad {,}0473^{\frac{3}{2}};$$

$$,00629^{\frac{3}{4}};\quad ,0005173^{\frac{4}{3}};\quad ,0092761^{\frac{5}{8}}.$$

All the observations which were made in regard to inverse integer powers apply to inverse fractional powers; thus,

$$\sqrt[5]{(7^{-3})} = 7^{-\frac{3}{5}} = 1 \div 7^{\frac{3}{5}};$$

so that the values of inverse fractional powers can also be computed.

EXAMPLES.

$$5^{-\frac{3}{4}};\quad 11^{-\frac{1}{2}};\quad 13^{-\frac{2}{7}};\quad 1{,}63742^{-\frac{2}{3}};$$

$$,018932^{-\frac{1}{4}};\quad ,000769^{-\frac{2}{5}};\quad ,897315^{-\frac{4}{5}}.$$

The multiplication, division, involution, and evolution of fractional powers differ in no essential respect from the multiplication, division, involution, and evolution of integer powers.

EXAMPLES.

$$5^{\frac{1}{2}} \times 5^{\frac{3}{2}};\quad 7^{\frac{2}{3}} \times 7^{\frac{5}{3}};\quad 11^{\frac{4}{5}} \div 11^{\frac{1}{5}};\quad 17^{\frac{5}{7}} \times 17^{-\frac{3}{7}};$$

$$6^{\frac{5}{3}} \div 6^{\frac{7}{3}};\quad 9^{\frac{4}{5}} \div 9^{-\frac{2}{5}};\quad 13^{\frac{2}{3}} \times 13^{\frac{3}{5}};\quad 4^{\frac{5}{7}} \times 4^{\frac{3}{5}};$$

$$8^{\frac{4}{11}} \times 8^{\frac{5}{12}};\quad (3^{\frac{2}{5}})^{4};\quad (7^{\frac{5}{3}})^{-3};\quad (6^{-\frac{1}{2}})^{2};$$

$$(5^{-\frac{3}{4}})^{-5};\quad (23^{\frac{4}{5}})^{\frac{3}{7}};\quad (13^{-\frac{2}{3}})^{\frac{3}{2}};\quad (23^{-\frac{3}{7}})^{-\frac{4}{3}}.$$

E. The most important property of numbers which the consideration of fractional powers unveils, is this, *that any one number may be regarded as a power of any other number.*

Thus the second power of the number 7 is 49, and its third power 343, so that, if we were to confine our attention to the integer powers of numbers, we might say that no number between 49 and 343 can be a power of 7; but when we come to use

fractional exponents, we see that as there is an infinity of fractions intermediate in value between 2 and 3, so there is an infinity of powers of 7 between its second and its third power.

The number 200, then, may be considered as a power of the number 7, having a fractional index between 2 and 3.

However, on attempting to realise this idea, we meet with a difficulty. No root of the number 7 is rational; that is, can be expressed with absolute precision by any fraction; and the same thing is true of all powers of the roots of 7, except those which come to be integer powers of 7 itself: this being the case, no fractional power of 7 can ever be exactly 200. Yet we can always find a fractional index, which may give the required value to within any prescribed degree of exactitude, just as we have already seen that although no fraction can accurately express the root of 13, we may yet find fractions to come as near to it as we may require.

This consideration leads us to *irrational* indices.

If we attempt to form an idea of what may be meant by $7^{\sqrt{2}}$, we are obliged to regard the index $\sqrt{2}$ as represented approximately by some fraction, as $\frac{7}{5}$; $7^{\frac{7}{5}}$ or $\sqrt[5]{(7^7)}$ is then an approximation to the value of $7^{\sqrt{2}}$. But $\frac{7}{5}$ is a very rude approximation to $\sqrt{2}$; we may use $\frac{17}{12}$, which is nearer, and may put $7^{\frac{17}{12}}$ for $7^{\sqrt{2}}$. In this way, by pushing our fractional exponent to a greater and greater degree of exactitude, we may obtain the value of $7^{\sqrt{2}}$ to any required degree of accuracy.

The labour attending these operations would be enormous, as any one may perceive, on attempting to compute the numerical value of the above, or of any similar expression.

CHAPTER XXIV.

ON THE NATURE AND COMPUTATION OF LOGARITHMS.

D. After the invention of the decimal system, no discovery in the science of numbers has proved so beneficial to the progress of knowledge as that of Logarithms. The memory of few men should be more cherished than that of John Nepair, their inventor.

In the preceding chapters I have endeavoured to lay before the student the doctrine of powers, in such a way as to prepare him for thoroughly comprehending the principles on which the logarithmic calculus is founded. The modern notation of powers, and the many minor improvements which have been made in the art of calculation, have rendered the task of explaining the nature and uses of logarithms comparatively an easy one. But in the days of John Nepair these facilities had no existence: it may be said, indeed, that they rather resulted from the invention of logarithms, than that the discovery was in any way facilitated by them. In order to form a true estimate of the peculiar originality and the high merit of the discovery, we must imagine ourselves deprived of all these modern helps.

In the preceding chapter I have given a method of extracting roots of all orders, which is, so to speak, new; since, though published twenty-seven years ago, it has, so far as I know, hardly been taught, except in my own class-room. Previously no method was known of extracting roots, the indices of which contain higher prime factors than the number three;

this method removes the last trace of difficulty from the doctrine of logarithms, and enables me to lay it clearly and readily before the student of ordinary arithmetic.

I would recommend the learner, before proceeding farther, to satisfy himself that he has fully apprehended the doctrines taught in the preceding chapters. Knowledge is built upon knowledge, and it is vain to expect to ascend the ulterior steps without having mounted the preparatory ones.

As already explained, we may regard any number, 200, as a power of any other number, 7. It is true that 200 is not one of the integer powers of 7, for these are 7, 49, 343, etc., but it may be represented to any required degree of precision by some fractional power; and hence a new kind of inquiry, "*To what power must* 7 *be raised in order to produce the number* 200?"

It is clear that 200 is above the second and below the third power of 7: the index of the power then must be 2 and a fraction. In order to find what this fraction is we may observe that for a change of unit in the index we have a change of 294, viz. from 49 to 343, in the power, and we may therefore, as a very rude guide in estimating the fraction, state the proportion

$$294 : 151 :: 1 : \tfrac{1}{2};$$

so that we may try $2\frac{1}{2}$ as the index of the power. On making the computation, we find

$$7^{\frac{5}{2}} = 129{,}64, \text{ etc.},$$

which is considerably below 200. A change of $\frac{1}{2}$ in the index of the power, that is, from $2\frac{1}{2}$ to 3, causes a change of 210 in the power, that is, from 130 (really 129,64) to 343; but we only need an augmentation of 70, and may therefore again use the proportion

$$210 : 70 :: \tfrac{1}{2} : \tfrac{1}{6},$$

whence the next trial may be made with the index $2^{\frac{2}{3}}$. On computing we find $7^{\frac{8}{3}} = 179{,}36$.

By proceeding in this way, toiling through the extractions of the roots, we might obtain the required index.

We may suppose, for the sake of illustration, that we have found the index to be 2,7228, that is, that

$$7^{2\cdot7228} = 200,$$

or, in other words, that the 27228th power of the 10 000th root of 7 is almost exactly 200.

Not only 200; any other number may be regarded as a power of 7: thus

$$7^{1\cdot4560} = 17\ ;$$

and we may imagine a table showing to what power 7 must be raised to give the series of numbers 2, 3, 4, ... 200, etc. Now the index of the product of two powers is the sum of the indices of those powers; therefore, since 3400 is the product of 200 by 17, we must have

$$7^{2\cdot7228} \times 7^{1\cdot4560} = 7^{4\cdot1788} = 3400\,.$$

And thus, by help of such a table, we could obtain the product of two numbers by adding the corresponding indices.

In such a table the number chosen as the root is called the *Basis* of the system, and the index of the power to which that basis must be raised in order to produce a given number is called the *Logarithm* of that number. Thus, in the above examples, 7 is the basis, 2,7228 is the logarithm of 200, 1,4560 the logarithm of 17, and 4,1788 the logarithm of 3400.

The word *logarithm* is compounded of the two Greek words λογος, logos, and αριθμος, arithmos, both of them signifying number, and the idea conveyed is that the logarithm of a number is the *number of ratios which it contains*; thus the ratio 2401 : 1 is compounded of four ratios 7 : 1, and the logarithm of 2401 is said to be 4 in reference to the basis 7.

From the above example it appears that the sum of the logarithms of two numbers is the logarithm of the product of those numbers; and since logarithms are the indices of powers of the same basis, the difference of the logarithms of two numbers is

the logarithm of the quotient; also the multiple of the logarithm of a number is the logarithm of the power of that number, and a fraction of the logarithm of a number is the logarithm of the corresponding root. Hence a complete table of logarithms would enable us to perform multiplication by addition, division by subtraction, involution by multiplication, and evolution by division.

Having assumed some number as the basis of the system, we have to discover to what power this basis must be raised in order to produce each one of the natural numbers. In order to explain clearly how this may be accomplished, I shall assume that

$$7^{2.7228} = 200,$$

or, converting the decimal into a common fraction, that $7^{\frac{226}{83}} = 200$, in other words, that 700 is the 226th power of the 83d root of 7. Having extracted the 83d root of 7, 7 is necessarily the 83d power, and 200 is the 226th power of that root.

If, then, I divide 200 by 7, I shall obtain the 143d power of the root, that is to say $28{,}571428 = 700^{\frac{143}{83}}$. Dividing again by 7, I obtain $4{,}081632 = 7^{\frac{60}{83}}$; but if I attempt now to divide by 7, I obtain a quotient less than unit on the one hand, and an inverse power on the other. From this it is seen that I can divide 200 as often by 7 as there are units in the value of the fraction $\frac{226}{83}$, that is, as often as I can take 83 out of 226.

Now if 4,081632, etc., be the 60th power of the 83d root of 7, 7 must necessarily be the 83d power of the 60th root of 4,081632, or

$$7 = 4{,}081632^{\frac{83}{60}}.$$

I may therefore divide 7 by 4,081632 as often as there are units in the fraction $\frac{83}{60}$, that is, as often as 60 goes in 83, the division being only carried on until the quotient be less than the divisor. If, then, I try how often 7 can be divided by 4,081632, I shall have ascertained how often the denominator

60 *should* be contained in the numerator 83. On dividing once we find the quotient $1{,}715$, so that

$$1{,}715 = 4{,}081632^{\frac{23}{60}},$$

and consequently $4{,}081632 = 1{,}715^{\frac{60}{23}}$.

Here, again, as often as $4{,}081632$ can be divided by $1{,}715$ without giving a quotient less than unit, so often *ought* the denominator 23 to be contained in the numerator 60. On dividing we find the quotient $2{,}379\ 961$, which is greater than the divisor; on dividing this we obtain $1{,}387\ 732$, wherefore the denominator 23 *ought* to be contained twice in 60; it is so, and therefore

$$1{,}387\ 732 = 1{,}715^{\frac{14}{23}},$$

or

$$1{,}715 = 1{,}387\ 732^{\frac{23}{14}}.$$

Proceeding in the same course, we divide $1{,}715$ by $1{,}387\ 732$, and obtain the quotient $1{,}235\ 829$, which is less than the divisor; we have here only one division, so that 14 ought to be contained once in 23 with a remainder, and

$$1{,}235\ 829 = 1{,}387\ 732^{\frac{9}{14}},$$

or

$$1{,}387\ 732 = 1{,}235\ 829^{\frac{14}{9}}.$$

Similarly, on trial, we find one division giving the quotient $1{,}122\ 916$, whence

$$1{,}122\ 916 = 1{,}235\ 829^{\frac{5}{9}},$$

or

$$1{,}235\ 829 = 1{,}122\ 916^{\frac{9}{5}}.$$

Again dividing, we have,

$$1{,}100\ 554 = 1{,}122\ 916^{\frac{4}{5}},$$

or

$$1{,}122\ 916 = 1{,}100\ 554^{\frac{5}{4}}.$$

Whence dividing

$$1{,}020\ 319 = 1{,}100\ 554^{\frac{1}{4}},$$

or

$$1{,}100\ 554 = 1{,}020\ 319^{4}.$$

Throughout this illustration, I have supposed that the fraction $\frac{226}{83}$ is truly the index of the power to which 7 must be raised in order to give 200; and hitherto we have seen nothing to contradict this assumption; but now, on dividing 1,100 554 by 1,020 319, we find the successive quotients

1,078 637,
1,057 157,
1,036 104,
and 1,015 471.

But if our assumption had been correct, the last of these quotients should have been unit.

Leaving this consideration for the moment, and looking back upon the process, we perceive that the number of divisions in each case necessarily coincides with the number of times that the denominator of the index can be taken out of its numerator; the numbers of those divisions, therefore, are just the quotients of the ordinary operation for finding the ratio of the numerator to the denominator of the original index. Now the numbers of these divisions, or *divisibilities*, as we may call them, can be easily ascertained, wherefore we can, in all cases, find the quotients whereby we may produce the Brounckerian series of fractions which approximate to that index.

The work may be concisely arranged as under. The first column contains the divisors, the second contains the dividends and quotients:

7,000 000 000	200,000 000 000
	28,571 428 571
2 Divisions.	4,081 632 653
4,081 632 653	7,000 000 000
1 Division.	1,715 000 000
1,715 000 000	4,081 632 653
	2,379 960 731
2 Divisions.	1,387 732 204
1,387 732 204	1,715 000 000
1 Division.	1,235 829 214
1,235 829 214	1,387 732 204
1 Division.	1,122 915 843

1,122 915 843 1 Division.	1,235 829 214 1,100 553 725
1,100 553 725 1 Division.	1,122 915 843 1,020 318 970
1,020 318 970 4 Divisions.	1,100 553 725 1,078 636 934 1,057 156 600 1,036 104 033 1,015 470 714
1,015 470 714 1 Division.	1,020 318 970 1,004 774 392
1,004 774 392 3 Divisions.	1,015 470 714 1,010 645 496 1,005 843 206 1,001 063 735
1,001 063 735 4 Divisions.	1,004 774 392 1,003 706 715 1,002 640 171 1,001 574 761 1,000 510 483
1,000 510 483 2 Divisions.	1,001 063 735 1,000 552 969 1,000 042 465
1,000 042 465 &c.	1,000 510 483 &c.

The quotients obtained in the Brounckerian operation for finding the ratio of the unknown index to unit are thus: 2, 1, 2, 1, 1, 1, 1, 4, 1, 3, 4, 2, etc., whence we have the converging series of fractions

$$\begin{array}{cccccccccccccc} & & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 4 & 1 & 3 & 4 & 2 \\ \frac{0}{1} & \frac{1}{0} & \frac{2}{1} & \frac{3}{1} & \frac{8}{3} & \frac{11}{4} & \frac{19}{7} & \frac{30}{11} & \frac{49}{18} & \frac{226}{83} & \frac{275}{101} & \frac{1051}{386} & \frac{4479}{1645} & \frac{10009}{3676}, \end{array}$$

so that, within a very great degree of precision, we have $7^{\frac{10009}{3676}} = 200$. That is to say, when 7 is assumed as the basis of a system of logarithms $\frac{10009}{3676}$, or in decimals, 2,7227965 is the logarithm of 200.

EXAMPLES.

What is the logarithm of 19 to the basis 2? Of 100 to the basis 11? Of 4381 to the basis 17?

Of the three, *the root, the index, the power,* we are now able from any two to compute the third. Thus if the root and the index be known, we find the power by involution; if the power and the index be known, we are able to extract the root; and if the root and the power be known, we can compute the index by the process just explained. Questions of the second class have hitherto only been resolved for square and cube roots, while questions of the third class have never been before attempted by common arithmetic. Now, however, all three are brought easily within our reach.

From this example it is obvious that the computation of a table of logarithms must be very laborious. Hence it becomes desirable to select, for the basis of operations, that number which may give the most useful results. It is not easy to adduce an argument in favour of one number rather than another, unless it be this in favour of the number TEN, that it is already universally adopted as the basis of the numeration scale, and that logarithms computed to the basis ten have peculiar advantages when expressed on the decimal scale. If we had been in the habit of counting by dozens and grosses, we should have found *twelve* to be the best basis for our logarithms.

Those views which led Nepair to the invention of logarithms, guided him to a system altogether independent of any numeration scale, and of which the basis is a particular ratio

$$2{,}71828\ 18284\ 59,$$

and it is singular that this system has since been found to possess characteristics so peculiar as to entitle it to be regarded as the natural system. But as soon as Nepair's logarithms were printed, it occurred to Briggs that a table computed to the

basis *ten* would be much more convenient in practice. On conferring with Nepair, Briggs found that the same idea had occurred to the inventor, and the adoption of the scheme resulted. The common or decimal logarithms are therefore distinguished by the name of Briggs.

Assuming *ten* as the basis of our system, let us endeavour to compute a table of logarithms.

The first number is 2, and we have to ascertain to what fractional power 10 must be raised in order to give 2.

Proceeding according to the method already described, and omitting the first column as redundant, we have the following calculation:

2,00000 00000 00000	
10,00000 00000 00000	
5,00000 00000 00000	
2,50000 00000 00000	
1,25000 00000 00000	3 Divisions.
2,00000 00000 00000	
1,60000 00000 00000	
1,28000 00000 00000	
1,02400 00000 00000	3 Divisions.
1,25000 00000 00000	
1,22070 31250 00000	
1,19209 28955 07812	
1,16415 32182 69348	
1,13686 83772 16160	
1,11022 30246 25157	
1,08420 21724 85504	
1,05879 11840 67875	
1,03397 57656 91285	
1,00974 19586 82895	9 Divisions.
1,02400 00000 00000	
1,01412 04801 82584	
1,00433 62776 61869	2 Divisions.
1,00974 19586 82895	
1,00538 23416 92974	
1,00104 15475 91550	2 Divisions.

1,00433 62776 61869	
1,00329 13020 22624	
1,00224 74136 42806	
1,00120 46113 91163	
1,00016 28941 37616	4 Divisions.
1,00104 15475 91550	
1,00087 85103 49749	
1,00071 54996 61433	
1,00055 25155 22278	
1,00038 95579 27960	
1,00022 66268 74155	
1,00006 37223 56541	6 Divisions.
1,00016 28941 37616	
1,00009 91654 62017	
1,00003 54408 47102	2 Divisions.
1,00006 37223 56541	
1,00002 82805 07155	1 Division.
1,00003 54408 47102	
1,00000 71601 37455	1 Division.
1,00002 82805 07155	
1,00002 11202 18476	
1,00001 39599 81066	
1,00000 67997 94924	3 Divisions.
1,00000 71601 37455	
1,00000 03603 40080	1 Division.
1,00000 67997 94924	
1,00000 64394 52524	
1,00000 60791 10253	
1,00000 57187 68113	
1,00000 53584 26102	
1,00000 49980 84221	
1,00000 46377 42470	
1,00000 42774 00848	
1,00000 39170 59357	
1,00000 35567 17995	
1,00000 31963 76763	
1,00000 28360 35661	
1,00000 24756 94689	

1,00000 21153 53847
1,00000 17550 13135
1,00000 13946 72552
1,00000 10343 32099
1,00000 06739 91776
1,00000 03136 51583 18 Divisions.

1,00000 03603 40080
1,00000 00466 88482 1 Division.

1,00000 03136 51583
1,00000 02669 63089
1,00000 02202 74597
1,00000 01735 81607
1,00000 01268 97619
1,00000 00802 09133
1,00000 00335 20650 6 Divisions.

1,00000 00466 88482
1,00000 00131 67832 1 Division.

1,00000 00335 20650

As this work proceeds the quotients become nearly equal to unit, exceeding it only by very small fractions; and on dividing, the new quotient is found to be *very nearly* unit augmented by the difference between the dividend and the divisor. When the number of zeroes preceding the fractional part is considerable, the figures of the quotient are found to agree with the figures of the difference for as many places as there are zeroes; and hence the division may be continued by simple subtraction whenever the first efficient figure of the divisor is at the middle place. It is therefore not necessary, in the present instance, to carry on the division farther. The number of divisions must be just the number of times that 131 67832 can be subtracted from 335 20650; so that the operation may now be completed by seeking the ratio of these two fractions. Hence the succeeding numbers are 2, 1, 1, 4, 1, 42, 6, etc., from which it appears that the ratio of unit to the logarithm of 2 is given by the successive quotients 3, 3, 9, 2, 2, 4, 6, 2, 1, 1, 3, 1, 18, 1, 6,

1, 2, 1, 1, 4, 1, 42, 6, etc., whence the successively approximative ratios

1		0
0	3	1
1	3	3
3	9	10
28	2	93
59	2	196
146	4	485
643	6	2136
4004	2	13301
8651	1	28738
12655	1	42039
21306	3	70777
76573	1	254370
97879	18	325147
1838395	1	6107016
1936274	6	6432163
13456039	1	44699994
15392313	2	51132157
44240665	1	146964308
59632978	1	198096465
103873643	4	345060773
475127550	1	1578339557
579001193		1923400330

These put in the decimal form give

Log. 2 = ,30102 99956 63981 19522,

which errs only by unit in the twentieth place.

For almost every business purpose logarithms to seven decimal places are sufficiently exact; so that if we carry the original calculations to ten places, and then shorten the results by leaving off the three last figures, augmenting the seventh figure when needed, we shall obtain all the precision that is required in practice.

Exercises.

Compute by the above method the logarithms of 3, 7, 11, and 13 true to ten places of decimals.

It is not necessary to go through the labour of this calculation for every number: it is enough to compute the logarithms of the primes since the logarithms of composite numbers can be got by adding together the logarithms of their factors. Thus from the logarithm of 2 we can readily obtain the logarithms of 4, 8, 16, 32, etc., as under:

Log. 2 = 0,30102 99957
Log. 4 = 0,60205 99913
Log. 8 = 0,90308 99870
Log. 16 = 1,20411 99827
Log. 32 = 1,50514 99783
Log. 64 = 1,80617 99740
&c. &c.

Also, since 10 is its own first power, the logarithm of 10 must be unit; since 100 is the second power of 10, the logarithm of 100 must be 2; that of 1000 3, and so on; thus:

Log. 10 = 1,00000 00000
Log. 100 = 2,00000 00000
Log. 1000 = 3,00000 00000
&c. &c.;

wherefore the logarithms of 20, being the sum of the logarithm of 10 and the logarithm of 2, must be

Log. 20 = 1,30102 99957
Log. 200 = 2,30102 99957
Log. 2000 = 3,30103 99957
&c. &c.

Again, since 5 is the quotient obtained on dividing 10 by 2, its logarithm must be the logarithm of 10, less the logarithm of 2; whence

Log. 5 = 0,69897 00043
Log. 25 = 1,39794 00086
Log. 125 = 2,09691 00129
&c. &c.

The next prime number is 3, and its logarithm, computed in the same way, is

Log. 3 = ,47712, &c.

From this we can find the logarithms of all numbers into which 2, 3, 5, enter as factors: those under 100 are,

4	12	20	30	45	64	80
6	15	24	32	48	72	81
9	16	25	36	50	75	90
10	18	27	40	60		

EXERCISE.

The student having computed the logarithms of 7, 11, and 13, may deduce those of the products of these primes with each other, and with the preceding numbers, as if he were making a table of the logarithms of all numbers under 100.

From the few examples which the student has worked out, he may readily understand that, to compute a table of the logarithms of all numbers up to say 100 000, is a task of enormous labour. By applying particular artifices, and taking advantage of our previous work, we may considerably reduce the labour. The process which Nepair used, was, perhaps, as tedious as that which I have given, and was much more circuitous in principle, yet the labour did not discourage the inventor. Of the method followed by Briggs I shall have occasion to treat shortly. But all these methods are practically superseded by others resulting from the doctrines of modern Algebra. So, satisfied that, in case of absolute need, we know how to construct a table of logarithms, I shall postpone the further consideration of this part of the subject, and proceed to show the use of logarithmic tables, as we find them already computed.

Neper, or, as he calls himself when writing in English, Nepair, published his logarithmic tables in 1614, under the title,

" Merifici
Logarithmorum
Canonis Descriptio.
Authore et Inventore
JOANNE NEPERO,
Barone Merchistonii
&c. Scoto."

This work is very rare: the logarithms therein given, besides being arranged only to suit calculations in trigonometry, were computed for that particular basis which I have mentioned; it is therefore very unlikely ever to be reprinted.

In the year 1624 Henry Briggs published a table of the common or decimal logarithms for all numbers up to 20 000, and for those between 90 000 and 100 000 computed to 14 decimal places. The work is entitled "ARITHMETICA LOGARITHMICA." And in the very title-page the author pays this tribute to the memory of the inventor:

"Hos numeros primus invenit clarissimus vir Johannes Neperus, Baro Merchistonij: eos autem ex ejusdem sententia mutavit, eorumque ortum et usum illustravit Henricus Briggius, in celeberrima Academia Oxoniensi Geometriæ professor Savilianus."

And in regard to the change of system, Briggs, while justly claiming the originality of the idea in his own mind, details the history of the change in the following words:

"Quod Logarithmi isti diversi sunt ab iis, quos Clarissimus vir Baro Merchistonii in suo edidit Canone mirifico, non est quod mireris. Ego enim cum meis auditoribus Londini, publice in Collegio Greshamensi, horum doctrinam explicarem; animadverti multo futurum commodius, si Logarithmus Sinus totius servaretur 0 (ut in Canone mirifico) Logarithmus autem partis decimæ ejusdem sinus totius, nempe sinus 5 graduum, 44, m. 21, s. esset 10 000 000 000. atque ea de re scripsi statim ad ipsum authorem, et quamprimum per anni tempus, et vacationem a publico docendi munere licuit, profectus sum Edinburgum; ubi humanissime ab eo acceptus hæsi per integrum mensem. Cum autem internos de horum mutatione sermo haberetur; ille se idem dudum sensisse, et cupivisse dicebat: veruntamen istos, quos jam paraverat edendos curasse, donec alios si per negotia et valetudinem liceret, magis commodos confecisset. Istam autem mutationem ita faciendam censebat, ut 0 esset Logarithmus unitatis, et 10 000 000 000 sinus totius: quod ego longe commodissimum esse non potui non agnoscere.

"Cœpi igitur ejus hortatu, rejectis illis quos anteà paraveram, de horum calculo serio cogitare: et sequenti æstate iterum profectus Edinburgum, horum quos hic exhibeo præcipuos, illi ostendi, idem etiam tertia æstate libentissime facturus, si DEUS illum nobis tamdiu superstitem esse voluisset."

Here Briggs tells his readers that, having imagined an improvement (actually consisting in the change of the basis), he communicated his idea to Nepair, and afterwards went to Edinburgh to confer with him. Nepair admitted the utility of the change, stating that he had long desired to make the calculations, if business and health had allowed, but that it would be better to make the logarithm of unit 0, and that of 10 unit. The advantage of this ulterior improvement, which at once freed the logarithmic system from trigonometry, was admitted; "so," says Briggs, "rejecting what I had prepared, I set to calculate these."

The table of Briggs, besides being incomplete, aspired to a degree of precision very far beyond our utmost wants; but it is a noble monument of zeal and perseverance.

In the years 1627 and 1628, Adrian Vlacq published at Gouda, the table of Briggs, completed for every number to 100 000; he judiciously restricted his logarithms to ten decimal places. This may be regarded as the first complete table of common logarithms. Vlacq modestly presents his work to the public with the very title-page of Briggs, merely adding, as if he were only editor, "Aucta per ADRIANUM VLACQ, Goudanum." And thus was completed, amidst the affectionate exchange of sentiments and good offices, one of the greatest and most beneficial inventions of modern times.

The arrangement of Vlacq's table is exceedingly simple, as may be seen from the subjoined specimen.

CHILIAS 3.

Num.	Logarithmi.	Differ.
30101	4.47858 09237	
		1 44277
30102	4.47859 53514	
		1 44272
30103	4.47860 97786	
		1 44267
30104	4.47862 42053	
		1 44262
30105	4.47863 86315	
		1 44258
30106	4.47865 30573	
&c.	&c.	&c.

It was soon found that even ten places of decimals far surpass the precision of the most accurate astronomical observations, and that seven-place tables would be amply sufficient for the wants of computers.

In arranging the seven-place logarithms, an improvement was introduced by John Newton, in his Trigonometrica Britannica, London, 1758, which is now universally adopted.

Instead of placing the logarithms of the successive numbers one below another, as in Briggs' and Vlacq's tables, John Newton placed them ten by ten in horizontal rows, so that the logarithms of all those numbers which end in one digit fall into one vertical column: also, instead of repeating the three first figures of the logarithms, he printed these only at the beginning of the line; thus,

	0	1	2	3	4	
3010	478 5665	5809	5954	6098	6242	&c.
11	7108	7252	7396	7540	7684	&c.
12	8550	8694	8838	8982	9126	&c.

By this arrangement the table is rendered more compact, and at the same time is more readily referred to.

The omission of the three first figures of the logarithms may occasion an error in the reading off, as may be seen on examining the subjoined extract:

	0	1	2	3	4	
2600	9733	9901	0068	0235	0402	&c.
01	415 1404	1570	1737	1904	2071	&c.

The logarithm of 26001 is 4 149 901, and that of 26 002 is 4 150 068 ; but as both the 9901 and the 0068 are in one line, we might be led to prefix 414 to the one as well as to the other.

In order to remove the chance of error arising from this ambiguity, Callet printed his Table Portatives with broken lines ; and Hutton in his later editions introduced a mark, called the change mark, above the first zero after the change. Many errors in calculation are to be traced to the non-observance of this change mark. In order to render it conspicuous, I introduced into Shortrede's tables a black zero (•) at the beginning of the change, and as far along the line as the zero is repeated.

Every computor ought to possess a copy of one or other of the seven-place logarithmic tables.

For purposes of less delicacy, Lalande published his five-place tables. These have been reprinted in London, and are sufficient for almost all business calculations, and for the purposes of ordinary land-surveying and engineering.

The computer must select that table which, being sufficiently accurate for the purpose in hand, contains the fewest extra figures, as otherwise he has much unnecessary labour.

When the logarithm of any number as 2 has been found, those of its multiples by the powers of 10 are very easily found ; thus :

Log.	2	=	,301 0300
Log.	20	=	1,301 0300
Log.	200	=	2,301 0300
Log.	2000	=	3,301 0300

or again,

Log.	16	=	1,204 1200
Log.	160	=	2,204 1200
Log.	1600	=	3,204 1200
Log.	16000	=	4,304 1200

And thus we see that the fractional part of the logarithm has, so to speak, to do with the figures of the number, while the integer part of the logarithm seems to regulate the place which

those figures occupy on the numeration scale. It is this circumstance which renders logarithms computed to the basis 10 preferable, in actual calculation, to logarithms suited to any other basis.

The logarithm of 30114 is 4,4787684, and from this we can obtain that of 301140 of 30,114 etc.: thus,

Log. 30114 = 4,478 7684
Log. 301140 = 5,478 7684
Log. 3011400 = 6,478 7684
Log. 3011,4 = 3,478 7684
Log. 301,14 = 2,478 7684

It is for this reason that the integer part of the logarithm is not given in the later tables: the fractional part or *mantissa*, as it is called, alone is printed. No inconvenience can arise from this omission, since the integer part can always be found by observing how many steps the highest figure of the number is removed from the unit's place.

The student should practise the seeking out of logarithms.

EXAMPLES.

Find, from any of the seven-place tables, the logarithms of the following numbers:—

861	296 8700	8,2228
4370	98,162	9,8407
3106	7,4311 194	,09
29761	509,38	9584,7
837,4	62,435	7163,2

Let us now consider the logarithms of fractions less than unit. We have seen that the inverse powers of numbers are less than unit: thus $3^{-2}=\frac{1}{9}$, $10^{-1}=\frac{1}{10}$. The logarithms, then, of fractions less than unit must have the sign $-$ prefixed, or, as we say, must be subtractive. In this way

Log. ,1 = $-$ 1,000 0000
Log. ,01 = $-$ 2,000 0000
Log. ,0001 = $-$ 4,000 0000

When we divide a number by 10 we lessen the logarithm by unit, since unit is the logarithm of 10. Hence, from the logarithm of 387, which is 2,587 7110, we obtain

Log. 38,7	=	1,587 7110
Log. 3,87	=	0,587 7110
Log. ,387	=	– 0,412 2890
Log. ,0387	=	– 1,412 2890
Log. ,00387	=	– 2,412 2890

Now the great convenience sought to be obtained by the adoption of the basis 10 is this, that the same mantissa belongs to all numbers consisting of the same figures: but this convenience is lost when we have to do with the logarithms of fractions less than unit: the mantissa then becomes the complement of what it was before, and thus it would seem that two complete logarithmic tables are needed.

However, we have seen that subtraction may be converted into addition by help of the arithmetical complement; and thus we may write

Log. ,387	=	9,587 7110
Log. ,0387	=	8,587 7110
Log. ,00387	=	7,587 7110

provided we recollect that the result obtained by adding these is 10 integers too great. By this arrangement the mantissa is preserved unchanged, but a dubiety is introduced, since 7,587 7110 is truly the logarithm of 38700000, while here it is used as the logarithm of ,00387; so that if we have to determine the number from its logarithm, we cannot distinguish which of these two it may be. The immense disparity of these numbers, however, prevents any practical uncertainty.

Examples.

Seek out the logarithms of the following fractions:—

,3	,00721	,0998	,000625
,047	,83743	,0000568	,00000439

The ordinary logarithmic tables give the logarithms of all five-place numbers, that is, of all numbers having five effective

figures: now we have often to work with numbers having more than five places, and we must therefore learn how to obtain their logarithms.

If we examine the differences between the logarithms of the successive numbers, we find that these differences decrease as the numbers increase: and it must necessarily be so. The logarithms, for example, of the two numbers 387 and 388 are 2,587 7110 and 2,588 8317, and the difference of these logarithms is 1 1207; but if I double each of them, the logarithms of these doubles, viz. of 774 and 776, must have exactly the same difference, since they are obtained by adding the logarithm of 2 to each of the above logarithms: the difference, then, between the logarithms of 774 and 776 must also be 1 1207; now between these two numbers there is the number 775, so that this 1 1207 must be divided into two minor differences nearly equal to each other. In point of fact,

$$\begin{array}{ll} \text{Log. } 774 = 2{,}888\ 7410 & \\ & 5607 \\ \text{Log. } 775 = 2{,}889\ 3017 & \\ & 5600 \\ \text{Log. } 776 = 2{,}889\ 8617 & \\ \hline & 1\ 1207 \end{array};$$

roughly speaking, we may say that the differences in the latter case are halves of the difference in the former case.

To take another illustration: the mantissæ of the logarithms of 3870 and 3880 are exactly the same as those of the logarithms of 387 and 388; wherefore the ten logarithmic differences between 3870 and 3880 must just make up the difference of the logarithms of 387 and 388. From these two examples we see that the difference between the logarithms of two adjoining numbers decreases nearly in proportion as the numbers increase.

But, not to take numbers so far apart, let us consider the three consecutive numbers 387, 388, 389. A change from 388 to 389 is not relatively so great a change as that from 387 to 388; the ratio 389 : 388 is not so high as the ratio 388 : 387. Now, the difference between the logarithms of two numbers depends on their relative disparity; wherefore the difference

between the logarithms of 388 and 389 must be less than the difference between those of 387 and 388. Accordingly,

Log. 387 = 2,587 7110	
	11207
Log. 388 = 2,588 8317	
	11179
Log. 389 = 2,589 9496	

Hence we see that the differences between the logarithms of numbers are not proportional to the differences between the numbers themselves. However, if we examine the differences of the logarithms given in page 109, we find that these differ among themselves by only 5 or 4 units in the tenth decimal place, and that, therefore, if the logarithms be cut short at the seventh place, as in our ordinary tables, no perceptible change takes place in their differences over a considerable range of numbers. Therefore, although the differences of the logarithms be not strictly proportional to the differences of the numbers, no practical error can arise from supposing them to be proportional from one five-place number to another.

We can thus deduce, with sufficient precision for all practical purposes, the logarithm of a six, seven, or eight-place number by a proportionate interpolation between the logarithms of the two contiguous five-place numbers.

Suppose that we wish to find the logarithm of the number 3010537 to ten places. On consulting the small extract given on page 109, and attending only to the mantissa or fractional part of the logarithm, we find

Log.	3010500 =	,47863 86315
Log.	3010600 =	,47865 30573
	100	1 44258

so that an increase of 100 in the number is accompanied by an increase of 144258 in the logarithm, and the increase for 37 in the number may be found by the proportion

100 : 37 :: 144258 : 53375 ;

whence the calculation—

Log. 3010500 =	,47863 86315
Increase for 37 =	53375
Log. 3010537 =	,47864 39690

But since the differences vary a little in the tenth decimal place, we cannot regard the last figure of this result as safe.

If we be satisfied with the precision given by seven-place tables, the calculation, conducted on exactly the same principles, becomes

$$\begin{array}{rl} \text{Log. } 3010500 = & ,478\ 6386 \\ \text{Log. } 3010600 = & ,478\ 6531 \\ \hline 100 & \qquad 145 \end{array}$$

hence 100 : 37 :: 145 : 54 and

$$\begin{array}{rr} \text{Log. } 3010500 = & ,478\ 6386 \\ \text{Increase for } 37 = & 54 \\ \hline \text{Log. } 3010537 = & ,478\ 6440 \end{array}$$

But here the result may be regarded as true to the very last place.

In order to facilitate such calculations, tables of proportional parts usually accompany the logarithmic tables; and these leave nothing farther to be desired in regard to the facility of inspection. The above logarithm would be found from Hutton's or Babbage's tables thus—

$$\begin{array}{rr} \text{Log. } 30105 \quad = & ,478\ 6386 \\ 3 \quad & 44 \\ 7 & 10 \\ \hline \text{Log. } 3010537 = & ,478\ 6440 \end{array}$$

After a little practice we are able to take out and add the correction mentally.

EXAMPLES.

Seek out the logarithms of the following numbers:

3,1415926	13,762531	7,623699
,43429448	490,7694	,81273588
365,242217	38273,672	4276,5183
8,360923	,006631753	10739,489
57,81039	,01863941	2186,4725

In the tables of proportional parts the results are given true to the nearest figure in the seventh place of the logarithm: they

are sometimes too great and sometimes too small: thus the proportional parts for the difference 137 are given.

1	14	whereas they should have been	1	13,7
2	27		2	27,4
3	41		3	41,1
4	55		4	54,8
5	69		5	68,5
6	82		6	82,2
7	96		7	95,9
8	110		8	109,6
9	123		9	123,3

And it may happen that, when there are two or three additional figures in the number, all the proportional parts may be too great, or all may be too small: in these cases the last figure cannot be correct. For example, if we take out the logarithm of the number 316,84219 from Hutton's tables, we have

Log. 316,84		= 2,500 8400
	2	27
	1	1
	9	1
Log. 316,84219		= 2,500 8429

But if we work out the proportion we find 1000 : 219 : : 137 : 30, so that the logarithm is truly 2,500 8430; therefore, when we wish the utmost degree of precision which the tables can give us, we must reject the auxiliary tables.

In Shortrede's Tables of Logarithms I caused the first eight multiples of the differences to be printed for the purpose of avoiding this source of minute error. The computation, by help of these tables, stands thus:—

Log. 316,84		= 2,500 8400
	2	27 4
	1	1 37
	9	1 233
Log. 316,84219		= 2,500 8430

It is, however, only in reference to matters which require the utmost degree of minuteness that such niceties need be attended to.

By help of the seven-place logarithmic tables we can obtain the logarithm of any number which occurs in ordinary practice. Let us now endeavour, by help of the same tables, to find the number corresponding to a given logarithm.

If the mantissa of the given logarithm agree with one in the table, the figures of the required number are at once found, and their position on the numeration scale is determined by the integer part of the logarithm. For example, if we wish to obtain the number of which 3,575 8111 is the logarithm, we observe that that number must be above the third power of 10, and below its fourth power; its highest figure, therefore, must be in the place of thousands. On searching in the table we find that the mantissa ,575 8111 corresponds to the figures 37654, and therefore we conclude that the required number is 3765,4.

EXAMPLES.

Find the numbers corresponding to the following logarithms :—

2,789 3409	0,238 8487	2,766 6508
3,846 0153	9,321 1635	1,856 0216
0,590 3513	7,663 5407	8,943 0392

But if the given logarithm be not found in the table, we examine the two logarithms nearest to it—the one above and the other below—and interpolate by proportion. Thus if it be required to find the number of which 2,168 5700 is the logarithm, we find

Log. 147,42 = 2,168 5564

Log. 147,43 = 2,168 5859.

The required number is, then, between 147,42 and 147,43.

The difference between these logarithms is 295, while the defect of the first of them from the given logarithm is 136;

wherefore if we desire to have two more decimal places we state the proportion

$$295 : 136 :: 100 : 46 ;$$

whence the number is 147,4246.

The tables of proportional parts enable us mentally to ascertain the subsequent figures: thus in difference-table 295 we find that 118 belongs to 4 as a next figure; this 118 subtracted from the defect 136 leaves 18; annexing a zero to this we have 180, and entering the table we find that 177 corresponds to 6; whence 4 and 6 are the subsequent figures.

$$\begin{array}{rl} 136 & \\ \underline{118} & 4 \\ 180 & \\ 177 & 6 \end{array}$$

EXAMPLES.

Find the numbers of which the following are the logarithms:—

3,463 0127	2,216 4871
1,602 7497	5,827 9173
8,581 0024	9,682 7342
7,063 4741	8,518 6372
0,323 7549	0,779 3625

ANTI-LOGARITHMS.

D. The tables of which I have hitherto been speaking give the logarithms of numbers by direct inspection, and the numbers to logarithms indirectly. In the year 1742, James Dodson published his Anti-Logarithmic Canon, in which the numbers are given corresponding to the successive logarithms. This table is the converse of the ordinary tables, and serves the same purposes with them. The manner of its formation is simple; and, what is worthy of remark, is in principle that which was followed by Briggs in his actual computations.

In order to compute the number which belongs to a given logarithm, we have to raise 10 to the power indicated by that logarithm: thus if we wish to discover that number of which ,379 is the logarithm, we must raise 10 to that power of which the index is ,379; that is to say, we must extract the thou-

sandth root of 10, and raise that root to its 379th power. And again, to compute that number which has ,29763 for its logarithm, we have to extract the hundred-thousandth root of 10, and raise that root to its 29763d power.

In point of fact, Dodson's table contains all the powers of the hundred-thousandth root of 10 up to its hundred-thousandth power.

In order to obtain the hundred-thousandth root of 10, Dodson extracted the tenth root, then the tenth root of that, and so on: but he gives no account of the process by which he extracted the tenth root, and it is to be inferred from some expressions in his description, that he had obtained the roots by the method of trial (probably assisted by the logarithmic tables), and had tested the accuracy by involving the assumed root to its tenth power. At that time no method was known for extracting other than second and third roots. The method which I have now given for the extraction of all roots, renders the formation of the anti-logarithmic canon exceedingly simple.

By extracting the square root of 10, we find

$$10^{\cdot 5} = 3{,}16227\ 76601\ 68379\ 33200;$$

and by extracting the fifth root of this, noting at the same time the 2d, 3d, and 4th powers of the root, we obtain

$$\begin{aligned}
10^{\cdot 1} &= 1{,}25892\ 54117\ 94167\ 21042\\
10^{\cdot 2} &= 1{,}58489\ 31924\ 61113\ 48520\\
10^{\cdot 3} &= 1{,}99526\ 23149\ 68879\ 60135\\
10^{\cdot 4} &= 2{,}51188\ 64315\ 09580\ 11109\\
10^{\cdot 5} &= 3{,}16227\ 76601\ 68379\ 33200
\end{aligned}$$

In extracting the fifth root to a great many decimal places, it is convenient to use separate sheets of paper rather than columns ruled on one sheet.

Multiplying $10^{\cdot 5}$ by $10^{\cdot 1}$, and continuing, we also find

$$\begin{aligned}
10^{\cdot 6} &= 3{,}91807\ 17055\ 34972\ 50770\\
10^{\cdot 7} &= 5{,}01187\ 23362\ 72722\ 85002\\
10^{\cdot 8} &= 6{,}30957\ 34448\ 01932\ 49434\\
10^{\cdot 9} &= 7{,}94328\ 23472\ 42815\ 02066\\
10^{1\cdot 0} &= 10{,}00000\ 00000\ 00000\ 00000
\end{aligned}$$

and the agreement of the last affords a proof of the accuracy of the work.

Extracting now the square root of $10^{\cdot 1}$ we obtain

$$10^{\cdot 05} = 1{,}12201\ 84543\ 01963\ 43559$$

and, taking the fifth root of this, noting the inferior powers, we have

$$10^{\cdot 01} = 1{,}02329\ 29922\ 80754\ 13097$$
$$10^{\cdot 02} = 1{,}04712\ 85480\ 50899\ 53346$$
$$10^{\cdot 03} = 1{,}07151\ 93052\ 37606\ 41741$$
$$10^{\cdot 04} = 1{,}09647\ 81961\ 43185\ 01314$$
$$10^{\cdot 05} = 1{,}12201\ 84543\ 01963\ 43559$$

By continued multiplication we raise this hundredth root of 10 to its hundredth power, and at each tenth power we have a test of the accuracy of the work, since these powers ought to agree with the powers of $10^{\cdot 1}$.

Proceeding in the same way we may obtain $10^{\cdot 001}$, and the student who wishes *thoroughly* to understand the subject can hardly do better than compute the value of this to (say) 15 places. Afterwards we can compute the values of $10^{\cdot 0001}$ and $10^{\cdot 00001}$. On raising this last root, step by step, to its hundred-thousandth power, we obtain the Anti-Logarithmic Canon of Dodson.

The extraction of the roots is as nothing compared with the labour of so many long multiplications. By attending to the progression of the differences, this labour may be very much abridged; but, with every appliance for lessening the labour, the task of constructing such a table is an arduous one.

Like all the early computers, Dodson aimed at excessive precision, and printed his results to eleven places, thus rendering his book far too cumbrous for common use. When Lieutenant (now Captain) Shortrede confided to me the arrangement of his Trigonometrical Tables, I resolved to include with them a reprint of Dodson's inverse table shortened to seven places. As my name does not occur in the second issue of Shortrede's tables, and is barely mentioned in the preface to the first issue, which was withdrawn from sale, I think it but justice to myself to

claim the whole of whatever merit may belong to the designing and arranging of the tables of logarithms and anti-logarithms, and of the Trigonometrical canon in that work.

The mode of using the inverse table does not differ from that of using the direct one. What is got from the one by direct inspection is got from the other by the converse process; and it may be remarked that the practised computer finds it fully more convenient to find a logarithm from Dodson's and a number from Hutton's table, than the other way.

Having now explained the mode of computing, arranging, and using logarithmic tables, I proceed to treat, in the next Chapter, of the application of Logarithms to various branches of calculation.

CHAPTER XXV.

ON CALCULATION BY MEANS OF LOGARITHMS.

D. LOGARITHMS were invented for the purpose of facilitating the operations of multiplication and division, particularly when the numbers are large. By far the greatest number of calculations in Trigonometry and Astronomy involve proportion; and as instruments came to be made more accurate, and astronomical knowledge to be more precise, the requisite multiplications and divisions grew to be so tedious, that the mere labour of computing threatened to put a stop to the farther progress of astronomy. It was for the purpose of removing this obstacle that John Nepair set himself to contrive some shorter process, and the result has been the invention of a method which has improved every branch of applicate arithmetic.

1.—ON MULTIPLICATION.

In order to multiply together two or more numbers by help of logarithms, we seek out the logarithms of the numbers, add them together, and having thus obtained the logarithm of the product, we seek out the number corresponding to that logarithm.

Thus if we have to multiply together the numbers 32,486, 517,732, 1,83747, and ,0397683, we proceed as under :

Log.	32,486	=	1,511 6962
Log.	517,732	=	2,714 1050
Log.	1,83747	=	0,264 2202
Log.	,0397683	=	8,599 5371
Log.	1229,019	=	3,089 5585

and obtain the product, 1229,019.

One of these factors is less than unit, and therefore its logarithm is subtractive, in reality – 1,400 4629; now instead of subtracting 1,400 5629, we have added its arithmetical complement; therefore it has been necessary to reject 10 from the amount.

Again, if we have to multiply together 38,07641, 1,836273, ,0914386, and ,00168973, we have the work thus:

Log.	38,07641	=	1,580 6560
Log.	1,836273	=	0,263 9372
Log.	,0914386	=	8,961 1296
Log.	,00168973	=	7,227 8173
Log.	,01080289	=	8,033 5401

In this case we ought to reject 20 from the amount; but we have only 18; so that, rejecting 10, the remaining 8 is, so to speak, 10 too much, and must therefore be the index of a number the highest figure of which occupies the place of hundredths; the product is then ,01080289.

The student may perform, by help of Logarithms, the exercises given in pages 69, 144, 145, of Volume I., as well as the following

Examples.

Multiply together 27,64, 19,73, 4,867, and ,03171, using five-place tables and also using seven-place tables.

An oblong block of stone is 37,56 inches long, 23,87 broad, and 25,07 thick, and one cubic inch of the stone is found to weigh 839,5 grains; required the weight of the stone.

Find the product of 5,87241, 7,36819, 13,42376, ,198743, ,201684, ,726891, and ,0290307.

A spring was found to supply 185,73 grains weight of water per second; how much does it supply in a year of 365,242 days?

The inhabitants of a town, 5763 in number, require 3,25 gallons of water each per day; what would need to be the contents of a reservoir to hold three months' supply?

What is the weight of a block of white marble 173,375

inches long, 107,625 broad, and 86,125 thick? The specific gravity of marble is 2,840.

What is the price of a wax-cloth 34 feet 2 inches long, and 23 feet 7 inches broad, at 2s. 11d. per square yard?

The hull of a ship measures 268,7 feet in length, 39,6 in breadth, and 14,3 in depth; and the displacement is reckoned at ,58 of the oblong; required the tonnage of the vessel, allowing 35 cubic feet to the ton.

What is the price of a sheet of plate-glass 8 feet 7,32 inches long, 5 feet 6,13 inches broad, and ,619 thick, at 7s. 3½d. per pound; the weight of a cubic inch of water being ,036036 lb., and the specific gravity of plate-glass being 2,642.

Required the continued product of 2,0; 1,9; 1,8; 1,7 . . . . . . 1,0.

To what does £3213 amount in 12 years, interest being at 3 per cent for the first year, and thereafter rising a quarter per cent every year?

What is the value of £9073 in the three-per-cents, the market rate being 91¾?

Required the value of £17635 of bank stock, at 183½.

2.—ON DIVISION.

As the logarithm of the product of two numbers is obtained by adding the logarithm of the multiplier to that of the multiplicand, so the logarithm of a quotient must be got by subtracting the logarithm of the divisor from that of the dividend.

If we have to divide the number 3972,56 by 483,972, we proceed thus:

Log.	3972,56	=	3,599 0704
Log.	483,972	=	2,684 8202
Log.	8,20824	=	0,914 2502

whence the quotient is 8,20824.

If the divisor be greater than the dividend, the logarithm of the quotient is subtractive; but we have no tables of subtractive logarithms, and therefore we take the arithmetical complement;

that is to say, we subtract as usual, merely adding 10 mentally to the logarithm of the dividend in order to render the subtraction possible. Thus to divide 7,862943 by 437,8244, we proceed as under :

Log.	7,862943	=	0,895 5852
Log.	437,8244	=	2,641 3000
Log.	,01795912	=	8,254 2852

obtaining the quotient ,01795912.

Again, if the divisor be less than unit, its logarithm is, according to usage, written 10 too much, so that the logarithm of the dividend must also be augmented by 10, unless the dividend also be less than unit. For example, the division of 31,6925 by ,0937843 is thus performed :

Log.	31,6925	=	1,500 9565
Log.	,0937843	=	8,972 1302
Log.	337,9297	=	2,528 8263

Perform the divisions given in pages 92, 149, 151, of Volume I., and the following

EXAMPLES.

A piece of copper wire 437,3 inches long weighed 5739,7 grains ; required the weight of one inch of the same wire ?

A sheet of tin of 370,2 square inches weighed 3059,3 grains ; what is the weight per square inch ?

The profits on a farm of 217,38 acres amounted to £473, 17s. ; required the rate of profit per acre ?

For £3763 what amount of stock can I purchase, the market price being 83$\frac{3}{4}$ per cent ?

In one year of 365,242217 days, the sun appears to describe the ecliptic ; what is its daily motion in ancient and in modern degrees ?

A cargo of corn cost in all £2943, and measured 971$\frac{1}{2}$ quarters ; what was the cost per quarter ?

Instead of subtracting the logarithm of the divisor, it is in general, though not always, preferable to add the arithmetical

complement of the logarithm, or the *cologarithm*, as it is called for shortness. Performed in this way, the preceding examples would stand as under :

Log.	3972,56	=	3,599 0704
Col.	483,972	=	7,315 1798
Log.	8,20824	=	0,914 2502

Log.	7,862943	=	0,895 5852
Col.	437,8244	=	7,358 7000
Log.	,01795912	=	8,254 2852

Log.	31,6925	=	1,500 9565
Col.	,0937843	=	1,027 8698
Log.	337,9297	=	2,528 8263

The student may apply this method to the former examples.

When multiplications and divisions are conjoined, as in simple and compound proportion, the use of the cologarithm is peculiarly advantageous.

Thus in order to compute the value of the fraction,

$$\frac{23{,}8693 \times 4{,}72694 \times {,}037685}{497{,}618 \times {,}0091734},$$

we should, in the usual way, add together the logarithms of the factors of the numerator, and also those of the factors of the denominator, and then subtract the sum of the latter from the sum of the former ; by using the cologarithms we avoid a considerable quantity of labour : thus,

Log.	23,8693	=	1,377 8397
Log.	4,72694	=	0,674 5801
Log.	,037685	=	8,576 1685
Col.	497,618	=	7,303 1039
Col.	,0091734	=	2,037 4697
Log.	,931455	=	9,969 1619

EXAMPLES.

Required the weight, in tons, of a block of granite 79,61 feet long, 17,97 thick, and 19,04 broad ; the specific gravity of the stone being 2,737.

A spring supplies 73,85 cubic inches of water per second ; how many gallons does this give per day to each one of a population of 5739 persons ?

The fall of rain being 37,639 inches during the year, what is the weight, in pounds, of the average daily fall upon an acre of ground ?

To what depth would one ton of water cover a square mile ?

What is the thickness of sheet lead which weighs 73,43 lb. per square yard ?

3.—PROPORTION.

The method of resolving a question in proportion is evident: for example, in order to find the fourth proportional to 73,8657, 51,3872, and 173,646, we proceed thus :

Col.	73,8657	=	8,131 5572
Log.	51,3872	=	1,710 8550
Log.	173,646	=	2,239 6648
Log.	120,8028	=	2,082 0770

A great convenience attending the use of the cologarithm in proportion is this, that the order of the statement does not need to be deranged.

The student may solve the exercises given in Volume I., pages 172, 173, 174, as well as the following

Examples.

Into a cask capable of holding 26,85 gallons, 17,43 gallons of spirits were poured, and the rest filled with water. Out of this another cask holding 10,58 gallons was filled: what quantity of pure spirit is in it ?

If the moon's motion in three hours be $1^\circ\ 25'\ 27''$, what is its motion in $2^h\ 53^m\ 27^s$?

If the moon's motion in three hours be $1^\circ\ 23'\ 28''$; in what time does it move through $51'\ 39''$?

Into a vessel containing 37,817 gallons of water, 253 grains of alum were put ; having stood long enough to allow the alum

to be dissolved, 971 qrs. of the solution were taken out and dried by evaporation. What weight of alum should have been found?

In order to examine the purity of water supplied to a town, 2739,7 grains of it were evaporated to dryness, when a residue weighing 1,684 grains was found. How much impurity was there in one gallon of the water?

The work for distributive proportion is greatly shortened by the use of a movable piece of paper. Simple mechanical contrivances of this kind often save us a great deal of writing.

Let us suppose that £39273,6 is to be distributed among the partners A, B, C, D, E, F, and G, in shares proportional to the numbers 38 927, 53 721, 86 943, 59 707, 20 692, 13 024, and 4876. We arrange the calculation as under:

A	38927	4,590 2509	3,740 4787	5501,470
B	53721	4,730 1441	3,880 3719	7592,275
C	86943	4,939 2346	4,089 4624	12287,468
D	59707	4,776 0253	3,926 2531	8438,264
E	20692	4,315 8025	3,466 0303	2924,356
F	13024	4,114 7444	3,264 9722	1840,654
G	4876	3,688 0637	2,838 2915	689,115
	277890	5,443 8729		39273,602
	£39273,6	4,594 1007		
		9,150 2278		

Having arranged all the arguments in one column, and taken their sum, we place their logarithm, as well as the logarithm of their sum, in the next column. Then from the logarithm of the quantity to be distributed we subtract that of the sum of the arguments. The remainder, in this case 9,150 2278, having been written on the edge of a slip of paper for the convenience of juxtaposition, is then added to the logarithms of the separate arguments, the sums being written in a third column; these sums are the logarithms of the shares belonging to the several parties; the shares themselves are thence found, and their sum is taken, in order to guard against error. In the present instance the sum of the shares exceeds the sum to be distributed

by the minute quantity ,002 ; this error is caused by the minute and unavoidable errors in the last places of the logarithms ; after all, it is only an error of two farthings among £39 000.

In addition to those given in Vol. I. p. 183, the student may solve the following

EXAMPLES.

An assessment of £2769, 18s. 3d. is to be levied according to the extent of the properties in a parish ; viz. belonging to A, 4738,3 ; to B, 693,7 ; to C, 2708,9 ; to D, 3091,7 ; to E, 53,2 ; to F, 187,6 ; to G, 931,2 ; to H, 417,6 ; to I, 1763,3 ; to K, 4,7 ; to L, 9,3 ; to M, 15,4 ; to N, 73,6 ; to O, 2,9 ; to P, 1,3 ; and to Q, 243,9. Required the amount to be paid by each one.

A company having realised a profit of £40 723, 17s. 6d., resolves to divide it according to the sums contributed by the partners : A's capital was £7973 ; B's, £6285 ; C's, £4379 ; D's, £2371 ; E's, £592 ; F's, £1683 ; G's, £217 ; H's, £3051 ; I's, £79 ; K's, £823 ; L's, £9308 ; M's, £7217 ; N's, 1991 ; O's, £4109 ; P's, £2367 ; Q's, £5684 ; R's, £12395 ; S's, £765 ; and T's, £7349. Required the share belonging to each.

4.—INVOLUTION.

In order to raise a number to any power, we have to multiply the logarithm of the number by the index of the power.

For example, let it be proposed to raise 1,379627 to its fourth power.

$$\begin{array}{llcl} \text{We have} \quad \text{Log. } 1{,}379\,627 & = & 0{,}139\,7617 \\ & \times & 4 \\ \hline \text{Log. } 3{,}622\,820 & = & 0{,}559\,0468 \end{array}$$

Whence the fourth power is 3,622 820.

The principle of this operation has already been explained ; but the student may observe that 1,379627 is that power of 10 which has the index 0,139 7617, or that

$$1{,}379627 = 10^{\cdot 1397617},$$

and that therefore the fourth power of 1,379 627 must be that power of 10 of which the index is ,559 0468.

It is to be remarked of this operation, that the result has not that degree of relative precision which belongs to the generality of operations with seven-place logarithms. The last figure 7 of the logarithm may stand for anything between 6,5 and 7,5 ; all that we learn from the table is, that the logarithm of 1,379 627 is between 0,139 76165 and 0,139 76175 ; so that the fourth multiple of this must lie between the limit 0,559 0466 and 0,559 0470. The fourth power, which we are seeking, may then, as far as our ordinary logarithmic tables tell us, be anything between the two limits

3,622818
3,622822

Of course the degree of uncertainty becomes the greater, the greater the index of the power.

Examples.

Compute the following powers, and the limits of the error in using seven-place tables,

$$5{,}23851^{5}; \qquad 37{,}0694^{7}; \qquad 1{,}73629^{9}.$$

When we have to compute the power of a quantity less than unit, we find some trouble in managing the index ; a little attention to the nature of the case may remove the difficulty. Thus, if we seek the fifth power of ,091732, and take its logarithm in the usual way, we have,

Log.	,091732	=	8,962 5209
		×	5
Log.	,000006495379	=	4,812 6045

Here the logarithm 8,962 5209 is regarded as 10 too high, so that the product 44,812 6045 is 50 too high ; we have then to reject 50 from the index, and only finding 40, we reject the 40, so as to leave the result 10 too much.

The same result is obtained when we use the true logarithm; thus,

$$\begin{array}{llcr}
\text{Log.} & {,}091732 & = & -1{,}037\,4791 \\
 & & \times & 5{,} \\
\hline
 & & & -5{,}187\,3955 \\
 & & \text{or} & 4{,}812\,6045
\end{array}$$

Examples.

Required the values of

$$\begin{array}{ccc}
{,}86391^{3}; & {,}71924^{4}; & {,}016897^{5}; \\
{,}99999^{6}; & {,}0023759^{3}; & {,}24738^{4}.
\end{array}$$

5.—COMPOUND INTEREST.

Logarithms give great facility to the calculation of compound interest. The mode of applying them is obvious.

Let the amount of £3879,35 in seventeen years at 4 per cent compound interest be required.

$$\begin{array}{llcr}
\text{Log.} & 1{,}04 & = & 0{,}017\,0333 \\
 & & \times & 17{,} \\
\hline
 & & & 0{,}289\,5661 \\
\text{Log.} & 3879{,}35 & = & 3{,}588\,7590 \\
\hline
\text{Log.} & 7556{,}575 & = & 3{,}878\,3251
\end{array}$$

As calculations of this kind are of frequent occurrence in business, and as the uncertainty in the last figures may cause inconvenience, it is proper to take the logarithm of the rate of improvement of money from more extended tables. Such tables are given by Hutton and Callet. The more exact calculation would stand thus:

$$\begin{array}{llcr}
\text{Log.} & 1{,}04 & = & 0{,}017\,0333\,393 \\
 & & \times & 17{,} \\
\hline
 & & & 0{,}289\,5667\,681 \\
\text{Log.} & 3879{,}35 & = & 3{,}588\,7590 \\
\hline
\text{Log.} & 7556{,}590 & = & 3{,}878\,3258
\end{array}$$

In addition to those given in Chap. XX. the student may solve the following

Examples.

To what does £5739 amount in 7 years, at $5\frac{3}{4}$ per cent, compound interest?

Required the amount of £3769, at 12 per cent per annum, in 13 years.

Required the amount of the same sum, at 6 per cent each half year; at 3 per cent payable quarterly; and at 1 per cent payable monthly; all in 13 years.

To what sum would £1 amount in 1857 years, at 3 per cent compound interest?

Required the amounts of £100 in 14 years, at 3, 4, 5, and 6 per cent respectively.

6.—INVERSE POWERS.

Since the logarithm of an inverse power is just the logarithm of the corresponding direct power, but made subtractive instead of additive, we have only to take the arithmetical complement of the logarithm of the direct power, and seek for the corresponding number.

For example, if $(7{,}364)^{-3}$ be required, we have

$$\begin{array}{lr}
\text{Log. } 7{,}364 & = 0{,}867\ 1138 \\
 & \times\ 3{,} \qquad\qquad \\
\hline
 & 2{,}601\ 3414 \\
\text{Log. } {,}00250414 & = 7{,}398\ 6586
\end{array}$$

Or,

$$\begin{array}{lr}
\text{Col. } 7{,}364 & = 9{,}132\ 8862 \\
 & \times\ 3{,} \qquad\qquad \\
\hline
\text{Log. } {,}00250414 & = 7{,}398\ 6586
\end{array}$$

Examples.

Required the values of

$18{,}373^{-1}$	$3{,}0436^{-4}$	${,}9634^{-1}$
$4{,}2961^{-2}$	$1{,}86273^{-5}$	${,}21769^{-2}$
$3{,}2917^{-3}$	$1{,}00824^{-3}$	${,}039874^{-3}$

7.—ANTICIPATED PAYMENTS.

The computation of the present value of a sum of money payable at some future time, is only a modification of the above process. Thus in order to compute the present value of £8967,7, payable 13 years hence, interest being allowed at 4 per cent, we have the following operation :—

$$\begin{array}{lcl}
\text{Col. } 1,04 & = & 9,982\ 9666\ 607 \\
 & & \times\ 13, \\
\hline
 & & 9,778\ 5666 \\
\text{Log. } 8967,7 & = & 3,952\ 6811 \\
\hline
\text{Log. } 5385,77 & = & 3,731\ 2477
\end{array}$$

In addition to those given in Chap. XXII. the student may solve the following

Examples.

What is the present value of £5907, payable 15 years hence, allowing interest at 3, at 4, at 5, or at 6 per cent ?

What is the present value of the reversion of a property worth £8960, let on lease for 999 years, of which 856 are to run, at the nominal rent of one barley-corn ; interest being at 3 per cent ? and what, interest being at 4 per cent ?

8.—ROOTS.

In order to find the logarithm of the root of a number, we must divide the logarithm of the number by the index of the root.

Thus to compute the seventh root of 89,635, we proceed as below.

$$\begin{array}{lcl}
\text{Log. } 89,635 & = & 1,952\ 4776 \\
 & & \div\ 7, \\
\hline
\text{Log. } 1,90075 & = & 0,278\ 9254
\end{array}$$

When we have to extract the root of a fraction less than unit, we must attend to the management of the index.

Thus if the 7th root of ,089635 be required, we observe

that the logarithm, as usually written 8,952 4776, is complementary, and that it is truly $-1,047\ 5224$; the work, then, would stand thus :—

$$\begin{array}{rl} \text{Log. } ,089\ 635 = & -1,047\ 5224 \\ & \div 7, \\ \hline & -0,149\ 6461 \\ \text{Log. } ,708\ 523 = & 9,850\ 3539 \end{array}$$

However, we can manage the calculation much more neatly by rendering the logarithm of the fraction 70 too much, in order that its 7th part may be 10 too much : now 8,952 4776 is already 10 too much ; hence

$$\begin{array}{rl} \text{Log. } ,089\ 635 = & 68,952\ 4776 \\ \div & 7, \\ \hline \text{Log. } ,708\ 523 = & 9,850\ 3539 \end{array}$$

EXAMPLES.

$\sqrt[5]{2897}$; $\sqrt[7]{689,71}$; $\sqrt[3]{1,02989}$;
$\sqrt[4]{,48735}$; $\sqrt[3]{3,14159}$; $\sqrt[5]{,28639}$;
$\sqrt[2]{,0086253}$; $\sqrt[7]{,0018973}$; $\sqrt[3]{,000897}$;
$\sqrt{2}$; $\sqrt[3]{3}$ $\sqrt[4]{4}$; $\sqrt[5]{5}$; $\sqrt[6]{6}$;
$\sqrt[7]{7}$; $\sqrt[8]{8}$; $\sqrt[9]{9}$; $\sqrt[10]{10}$.

The question, "At what rate must interest be in order that a certain sum of money may amount to so much in so many years ?" is one belonging to this subject.

£4723 was embarked in a concern for 23 years, and at the end of the time was found to have produced £16791 ; required the average rate of interest.

$$\begin{array}{lrl} \text{Log. } 4723 & & = 3,674\ 2179 \\ \text{Log. } 16791 & & = 4,225\ 0766 \\ \hline & 23 & \big|\ 0,550\ 8587 \\ & & \big|\ \quad 91\ 1 \\ & & \big|\ \quad 21 \\ \hline \text{Log.} & 1,056697 & = 0,023\ 9504 \end{array}$$

so that interest must have been at 5,6697, or about $5\frac{2}{3}$ per cent.

EXAMPLES.

A sum of money, £3720, invested 18 years ago in a concern,

has been found to amount to £8965; required the average annual rate of interest.

What must be the annual rate of interest in order that a sum of money may be doubled in 20 years?

9.—FRACTIONAL POWERS.

The application of logarithms to the computation of fractional powers is readily made. We have only to multiply the logarithm of the number by the fractional index. Thus to compute the value of $37,862^{\frac{3}{5}}$ we proceed as below.

$$\begin{array}{lr} \text{Log. } 37,862 = & 1,578\,2036 \\ & \times\ 3, \\ & 5\,|\,4,734\,6108 \\ \text{Log. } 8,849572 = & 0,946\,9222 \end{array}$$

When the power of a fraction less than unit is required, we must observe that the real logarithm is subtractive; and if we use the ordinary complementary logarithm we must be careful in the management of the indices. Thus if the value of $,0089635^{\frac{4}{7}}$ be required, we may proceed in either of the following ways:

$$\begin{array}{lr} \text{Log. } ,0089635 = & 7,952\,4776 \\ \text{Or, } & -\,2,047\,5224 \\ & \times\ 4, \\ & 7\,|\,-\,8,190\,0896 \\ & -\,1,170\,0128 \\ \text{Log. } ,06760632 = & 8,829\,9872 \end{array}$$

Or,

$$\begin{array}{lr} \text{Log. } ,0089635 = & 7,952\,4776 \\ & \times\ 4, \\ & 7\,|\,61,809\,9104 \\ \text{Log. } ,06760632 = & 8,829\,9872 \end{array}$$

Here the first logarithm has been augmented by 10, wherefore its product by 4, viz. 31,809 9104, is 40 too much; but we wish it to be too great by 70, in order to prepare for division by 7; therefore we add 30 to it, thus making 61,809 9104, and then the quotient by 7 agrees with usage.

Examples.

$53^{\frac{2}{7}}$;	$,02697^{\frac{4}{3}}$;	$,071635^{-\frac{5}{7}}$;
$179^{\frac{5}{3}}$;	$,0002761^{\frac{2}{11}}$;	$,01973^{-\frac{7}{3}}$;
$18,672^{\frac{3}{4}}$;	$49,635^{-\frac{4}{5}}$;	$,6791^{-\frac{5}{2}}$.
$5,6971^{\frac{3}{5}}$;	$5,1729^{-\frac{2}{3}}$;	
$,8273^{\frac{2}{3}}$;	$,91632^{-\frac{3}{4}}$;	

10.—EXPONENTIAL PROBLEM.

Lastly, logarithmic tables may be employed to solve the primary question, " To what power must one number be raised in order to give another number ? "

Thus if it were asked, What power is 3897 of 7, we would inquire by what number the logarithm of 7 must be multiplied in order to give the logarithm of 3897 ; in other words, we must divide the logarithm of 3897 by the logarithm of 7 ; hence the work

$$\begin{array}{lll} \text{Log. } 7 = 0,845\ 0980 ; & \text{Col.} & = 0,073\ 0928 \\ \text{Log. } 3897 = 3,590\ 7304 ; & \text{Log.} & = \underline{0,555\ 1827} \\ & \text{Log. } 4,24889 & = 0,628\ 2755 \end{array}$$

Here we have taken the logarithms of the logarithms in preference to the operation by common division, which would have been

$$\frac{3,590\ 7304}{,845\ 0980} = 4,24889,$$

and the result is, that

$$7^{4{\cdot}24889} = 3897.$$

Examples.

For how many years must £2700 remain at $3\frac{1}{2}$ per cent compound interest, in order that it may amount to £6000 ?

In how many years is a sum of money doubled, at 3, 4, 5, 6, and 7 per cent ?

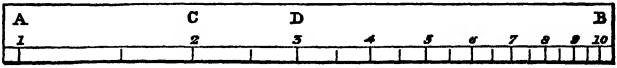
The number 4,24889 is the logarithm of 3897 in that system of which the basis is 7; and hence we conclude that, in order to convert logarithms suited to the basis 10 into logarithms suited to the basis 7, we must divide them by ,845 0980; and thus it seems that the logarithms in one system are proportional to the logarithms in another system.

11.—ON LOGARITHMIC LINES.

We have just seen that, in order to convert logarithms computed to the basis *ten* into logarithms suited to the basis *seven*, we must divide them all by ,8450980, or, what comes to the same thing, multiply them by the inverse of this, 1,183058; and that therefore logarithms in one system are proportional to logarithms in any other system. We may make our system such that the logarithm of 10 may be unit, that the logarithm of 7 may be unit, or that the logarithm of any other number may be unit; and the number which has unit for its logarithm is called the basis of the system.

Instead of representing logarithms by numbers we may represent them by lines, or by any other magnitudes. Let, for example, the line AB be taken to represent the logarithm of 10; then, if we make AB : AC : : log. 10 : log. 2, AC represents the logarithm of 2, and in order to obtain AC we may use the logarithm of Briggs or of Nepair indifferently; or, if we have them computed, the logarithm according to any other system. Since we have the decimal logarithms ready computed, we shall use them. Now, in our tables, the logarithm of 10 is unit, and the logarithms of other numbers are given in decimal parts of this unit; therefore it is most convenient for us to divide AB into 10, 100, 1000, etc., equal parts, and, having made a scale of those parts, to lay off by its help the logarithm of 2, viz. 301-thousandths of AB. Or, if we have ready divided scales of equal parts, it is as well to make AB 1000 from our scale, and AC 301. If AD be made 477, AD represents the logarithm

of 3, and so in this way we may represent geometrically the

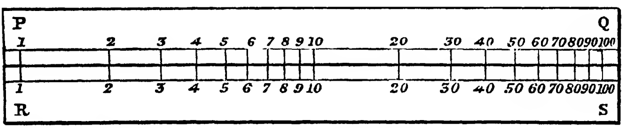


logarithms of all numbers, and may thus form what is called a *logarithmic scale* or *line.*

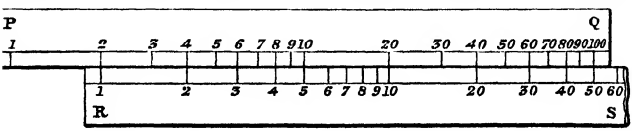
Such scales enable us very readily to make computations in proportion: they are manufactured in wood, brass, ivory, etc., and are known by the name of sliding rules, because they are made in pairs, the one to slide upon the other; and when carefully constructed they are valuable aids to the computer.

The difference between the logarithm of 10 and that of 100 is just equal to the difference between the logarithm of 1 and of 10; hence the numbers from 10 to 100 are crowded into the same room as those from 1 to 10; so that if we put in the logarithms of 1·1; 1·2; . . . . 5·1; 5·2; . . . . 9·9; in the first part of the scale its two halves exactly resemble each other.

Let, then, PQ and RS be two scales opposite to each other and prepared in this way. If we slide RS along until the be-



inning of it, viz. at 1, come opposite to the number 2 on the cale PQ; 2 on RS must be opposite 4 on PQ; 3 on RS pposite 6 on PQ; that is, the numbers on PQ are the doubles

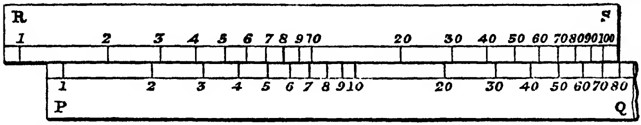


of those on RS. And this necessarily results from the nature of the scale: we have added the logarithm of 2.

Again if we bring the number 3 on RS opposite to the number 5 on PQ, any other number on RS opposite a number on

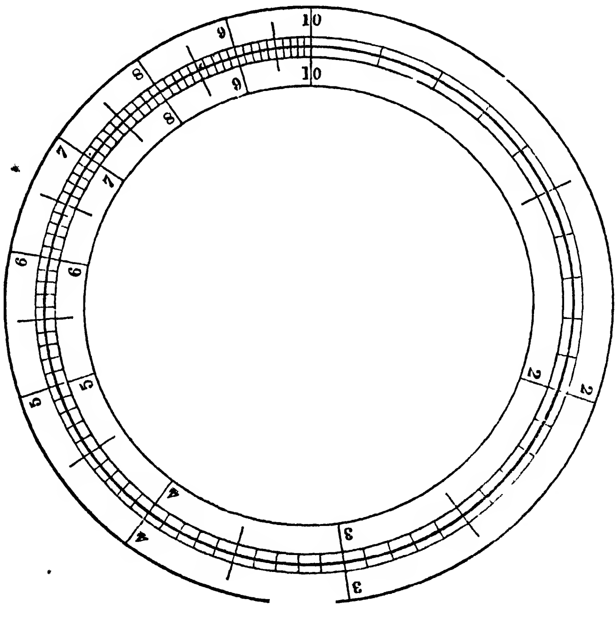
PQ is to that number in the ratio of 3 : 5 ; that is, 9 on RS is opposite 15 on PQ, and so on.

Hence, in order to work a proportion by help of the sliding rule, we bring the antecedent on the one scale, say PQ, opposite to its consequent on RS ; then the other antecedent on PQ is opposite its consequent on RS. For example, if we have to find a fourth proportional to 37, 52, and 49, we bring 37 on PQ



opposite 52 on RS, then opposite to 49 on PQ we find 68.9, which is the fourth proportional required.

A much more convenient form for the logarithmic scale is the circular. Having prepared two rings to turn, the one within



the other, we assume a whole turn as the logarithm of ten, and lay off in the proper proportions the logarithm of 2,3, . . . etc. up to 10; after this the logarithms are again marked off, 11, 12, . . . and the logarithm of 20 coincides with the logarithm of 2, that of 30 with that of 3, and so on; so that the mark 2 may be read 2, 20, 200, 2000, according to the number of complete turns which are supposed to have been made. The parts laid off on the circumference are thus, so to speak, the mantissæ or decimal parts of the logarithms; the integer parts have to be supplied by the computer.

The mode of using this instrument is the same as that of using the sliding rule.

The circular logarithmic scale is of great use to the practical computer in interpolating tabular differences, and most particularly in systematic calculations is its advantage felt. If at all well made it may be depended on to three figures.

The idea of representing logarithms by straight lines is due to Edmund Gunter, and was made public in 1623. Scales, with logarithmic lines engraven on them, pass therefore, very properly, under the name of Gunter's scales. In Gunter's arrangement compasses have to be used. Wingate, in 1627, used separate rulers, sliding the one against the other; and in the same year Oughtred contrived the circular form.

The only materials which we can well use for circular logarithmic scales are metallic, and the glitter of the metal causes inconvenience in reading off. A durable material having a dull surface is very much to be desired for this and similar purposes.

CHAPTER XXVI.

ON ARITHMETICAL SYSTEMS GENERALLY.

D. Having given, in the preceding chapters, a full account of the Decimal System of Arithmetic, and shown the methods of applying it to calculation, I now proceed to consider, shortly, the characters and properties of such systems in the abstract. Much useless discussion has arisen on the question "How came the number ten to be adopted as the basis?" and perhaps the answer that ten is the number of fingers on both hands is a sufficient one. But another question remains behind, "Is ten the best basis that could have been adopted?" the answer to which is not so easily obtained: it can only be obtained by an attentive study of the facilities which other bases might have afforded. It must not be forgotten that the sexagesimal and duodecimal systems obtained considerable extension, and that both remain in use to the present day. Of the one we have examples in the division of hours and degrees into minutes, seconds, thirds; of the other, we have a familiar instance in the division of the foot into inches and lines; while many contend that the duodecimal is preferable to the decimal scale.

The first work in which the various possible numeration scales were seriously treated was Leslie's *Philosophy of Arithmetic*, a work which has been eminently useful in leading us from the mere manipulation, which forms the staple of too many treatises on arithmetic, to the consideration of the principles which are involved.

BINARY SCALE.

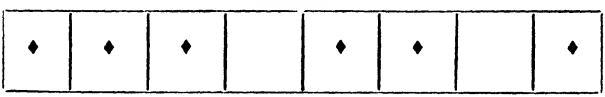
D. The mode of counting in twos had been very generally adopted: we have it in the subdivision of the gallon, of the inch as used by artisans ; of the yard as used by mercers. The division of the pound avoirdupois into 16 ounces is another example: and the *diram* of the East is divided by repeated halving into 1024 equal parts.

This scale of numeration is slow, and requires a perplexing multitude of names. While three steps of the denary system carry us to one thousand, no less than ten steps of the binary system are needed to take us about as far ; so that a binary abacus must have a great many columns.

In order to adapt the abacus to the binary system of numeration, we must suppose a counter in the second groove to stand for *two*, one in the third groove to stand for *four*, and so on, the value of the counter being doubled at each remove.

Thus the number indicated by the subjoined marks on the binary abacus is one, and four, and eight, and thirty-two, and

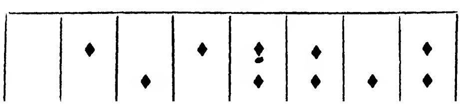
BINARY.



sixty-four, and one hundred and twenty-eight, or in all, two hundred and thirty-seven.

The different arithmetical operations may be performed on this abacus just as upon the decimal abacus ; thus to add to-

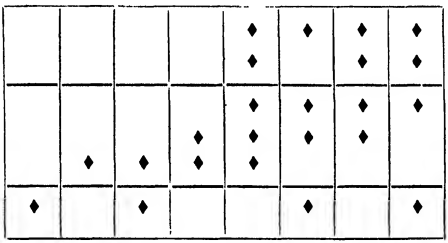
BINARY.



gether the two numbers represented below, we remove two counters whenever found in one grove, and for them place a single counter in the groove above, as shown in the example: and we can hardly doubt that, if we had been accustomed to it, the work would have been very easy, while simplifications would have been detected.

Since there ought never to be more than one counter in a groove, multiplication on the binary abacus requires only trans-

BINARY.



position. Thus the multiplication of *fifteen* by *eleven* would be performed on the binary abacus as shown in the annexed figure.

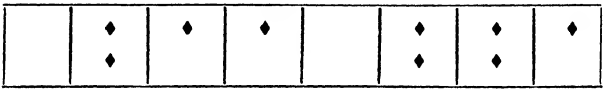
A greater number of grooves is needed on the binary abacus than on any other, but on the other hand, a smaller number of counters is required. Thus with nine counters we can mark on the binary scale every number up to one thousand six hundred and thirty-five, while with nine counters on the denary scale we can only get to eighteen. It is this circumstance which has mainly tended to keep the binary system so long in use, particularly for weights; for by having our weights successively doubled, we are enabled to weigh each unit up to the sum of them all. This advantage, however, by no means compensates for the slowness of the progression.

TERNARY SCALE.

It seems never to have been customary to count in *threes;* only in a very few instances are goods made up in parcels of three

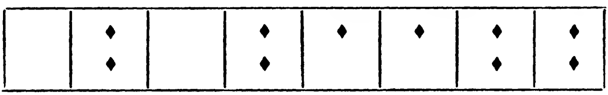
each. Were we to arrange an abacus on the ternary system, each remove would augment the value of the counter threefold, and no more than two counters would be needed in one groove. In this way a counter in the second place would stand for *three*, a counter in the third groove for *nine*, one in the fourth for

TERNARY.

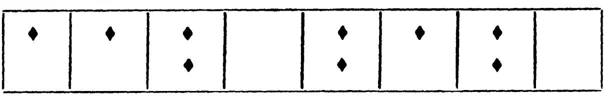


twenty-seven, and so on, and thus the subjoined example would represent *twice* 729, 243, 81, twice 9, twice 3, and 1 ; in all, *one thousand eight hundred and seven.*

Addition is quite readily performed on the ternary abacus : whenever we find three counters in one groove we remove them, placing one counter in the higher groove ; thus if to the above number we have to add this one,



the amount is



Multiplication also is quite as easily performed ; we have only to recollect that by removing all the counters of a number one step to the left, the number represented by them is augmented three fold.

The ternary is more rapid than the binary progression, and requires fewer columns, though more counters. For weighing, however, it possesses a curious though not a very practical advantage over the binary scale. If, in weighing, we be allowed to place weights in the pan along with the goods to be weighed, a set of weights in the progression 1, 3, 9, 27, 81, etc.

enable us to weigh with the smallest possible number of them; thus with only *seven* weights in this progression we are able to represent every number up to 1093. To weigh two ounces, we put the three ounce weight in one pan, and one ounce in the other pan of the balance. Representing the subtractive weights by rings (o) on the abacus, the subjoined may be taken as an ex-

♦	o	♦	o	o	♦	♦	o

ample; the number represented by it is $2187 - 729 + 243 - 81 - 27 + 9 + 3 - 1 = 1604$. The student may amuse himself by making up the representations of the successive numbers in this way. It is clear that the trouble of reckoning the values does away with any speculative advantage.

After these examples it is not worth while to enter into the details of other systems, particularly as the principles are the same throughout; it may be more profitable that I should point out the general features of the manipulations as applied to all.

Let it be proposed to represent the number of articles in a given heap upon any numeration scale, making use of the abacus. The number of grains in a quantity of rice is to be represented on the Septenary scale.

Taking out the grains in *sevens*, let us lay aside a grain or other counter for each seven, and continue this process until there be not so many as seven grains left; the number of counters laid aside is the number of sevens, and that of the grains left, the number of units: say that there are *three* grains left, we put three counters in the units' groove of the abacus.

Again taking out the counters in *sevens*, we lay aside a counter for each group, and place the remaining counters in the second groove: each of these stands for seven, and each counter of the second class for *forty-nine* grains; if there be one counter over, we put one counter in the second groove. Again taking out the counters in sevens, we find (say) twenty-six groups (for which

we put twenty-six counters), and one over; this one we place in the third groove. Lastly, counting out the twenty-six counters in sevens, we find three groups and five counters over, and therefore place five counters in the fourth, and three counters in the fifth groove, thus obtaining in all the subjoined representation, showing that the number of the grains of rice is three

SEPTENARY.

			♦ ♦ ♦	♦ ♦ ♦ ♦ ♦	♦	♦	♦ ♦ ♦

two-thousand four-hundred-and-ones, five *three-hundred-and-forty threes*, one *forty-nine*, one *seven*, and *three*.

This process is exactly the same as that for taling on the decimal abacus, and although the final statement of the result appear tedious, that is only because our nomenclature, suited to the denary scale, is not suited to the septenary one.

If the number be already expressed on the decimal scale, we can transpose it to any other scale by following the same plan. Thus, in order to mark the number 8977 upon the nonary scale, we take the *nines* out of it, and find 997 nines, with 4 over; so we put four counters in the units' groove of the abacus. Again we tale out the 997 in nines, and find 110, with 7 over; placing seven counters in the second groove, we proceed again in the same way as shown in the margin, and obtain,

9 | 8977
9 | 997...4
9 | 110...7
9 | 12...2
1...3

NONARY.

			♦	♦ ♦ ♦	♦ ♦	♦♦♦♦ ♦♦♦	♦♦♦♦

This operation, though different in appearance, is identical in principle with that of taling out the actual counters.

Examples.

Express the number one million on the binary, ternary . . . . nonary scales.

The converse problem, to transpose a number from any other to the denary scale would be performed in the very same way if that other scale were familiar to us. Actually the transposition may be effected in either of two ways. Thus the following number, as marked on the Duodenary scale, may be reckoned

Duodenary.

		♦ ♦	♦ ♦ ♦	♦	♦♦♦♦ ♦♦♦♦		♦♦♦♦ ♦

upon the denary scale, by computing the successive powers of the number 12, and taking the proper multiples of these; thus,

$$\begin{aligned}
2 \times 248832 &= 497664 \\
3 \times 20736 &= 62208 \\
1 \times 1728 &= 1728 \\
8 \times 144 &= 1152 \\
5 \times 1 &= 5 \\
\hline
&\ 562757
\end{aligned}$$

or the reduction may be effected, step by step, in this way.

Two counters in the sixth groove are equivalent to 24 counters in the fifth, thus making, with the three already there, 27 counters in the fifth groove; and so on, as under,

```
2...3...1...8...0...5
   27
      325
         3908
            46896
               562757
```

These processes for conversion are the very same in principle with those used for the reduction of compound quantities (Chap. XV.); they are more simple, inasmuch as the scale of progression is uniform.

By continuing the decimal notation downwards below the units' place, we indicate fractions of which the denominators are 10, 100, 1000, &c. So by continuing the abacus below the units' place, we may represent fractions of which the denominators are the successive powers of the basis of the scale. Thus, using a strong bar to separate the fractional from the integer part of the abacus, the following notation on the Quaternary

QUATERNARY.

	♦ ♦ ♦	♦	♦ ♦	♦	♦ ♦	♦ ♦	♦ ♦ ♦

scale would stand for,

$$3.16 + 1.4 + 2 + \tfrac{1}{4} + \tfrac{2}{16} + \tfrac{2}{64} + \tfrac{3}{256}\,;$$

that is, for $54\tfrac{107}{256}$.

In order to express a fraction upon any abacus, we must proceed exactly as we did for decimal fractions. Thus to mark the fraction $\tfrac{7}{13}$ on the Quinary scale, we observe that 7 units make 35 fifths; that is, 35 on the first fractional groove: the 13th part of this is 2 fifths, with 9 fifths over. These 9 fifths are equivalent to 45 twenty-fifths; that is, to 45 on the second fractional groove, and the 13th part of this is 3, with 6 over. Continuing this process we arrive at the following expression on the Quinary abacus for the fraction $\tfrac{7}{13}$. In this the terms

QUINARY.

	♦ ♦ *	♦ ♦ ♦	♦ ♦	♦ *	♦ ♦	♦ ♦ ♦	♦ ♦

recur in periods of four each, and are analagous to those of a circulating decimal fraction.

EXAMPLES.

Express the fractions $\tfrac{7}{18}$, $\tfrac{3}{5}$, $\tfrac{17}{23}$, upon the binary, ternary, ... nonary abacus.

The operations of addition, subtraction, multiplication, and division are performed on any of the scales exactly as they are performed on the denary scale ; but as we are accustomed to the latter, and not to the others, we find the actual work to be irksome, chiefly because we have to convert and re-convert the numbers with which we are working from one scale to the other. If we had been accustomed to any other scale, the denary would then have appeared as troublesome as the others now appear to us.

As it is tiresome to mark down the numbers of the counters one by one, we may shorten the work by using figures, and may write the above fraction.

QUINARY.

	2	3	2	1	2	3	2

or omitting the bars, and using ˛ as the mark for separating the fractional from the integer part.

QUINARY.

0˛232123212321, etc. ;

and by this means we obtain, on any scale, all the compactness which has been reached on the usual denary scale. So long as the basis of the scale is less than 10, the figures which we already have are enough ; but for scales to a basis above 10 it is necessary to adopt additional figures. Thus to write on the duodenary scale we need figures for the numbers *ten* and *eleven*. However, as we do not mean to practise on any scale, but merely to acquire more just and extensive ideas of numeration in general, we shall, for all scales but the denary, retain the bars which separate the different ranks. The above fraction may then be written thus :

QUINARY.

0	2	3	2	1	2	3	2	1	2	3	2	&c.

the strong bar serving to separate the fractional from the integer part.

Let it be required to transpose 73,843, 29,617, and 82,074 to the septenary scale, and then to add them together.

The most convenient mode of changing the fractional parts is to multiply them repeatedly by 7, transferring the integer part of each product to the septenary scale; thus

,843	,617	,074
5,901	4,319	0,518
6,307	2,233	3,626
2,149	1,631	4,382
1,043	4,417	2,674
0,301	2,919	4,718
2,107	6,433	5,026
0,749	3,031	0,182
&c.	&c.	&c.

The representatives on the septenary scale are given on the first three lines, and their sum on the fourth line of the accompanying figure.

SEPTENARY.

1	3	3	5	6	2	1	0	2	1	&c.
	4	1	4	2	1	4	2	6	3	&c.
1	4	5	0	3	4	2	4	5	0	&c.
3	5	3	3	5	1	1	0	6	4	&c.

EXAMPLES.

Express and add together the fractions $\frac{7}{8}$, $\frac{3}{5}$, $\frac{11}{13}$, and $23\frac{6}{11}$ on the senary, the septenary, and the nonary scales.

Express and collect 13,879 + 11,623 – 17,891 on the quinary and senary scales.

Again, let it be proposed to multiply 71,602194786 by 13,283950617 on the nonary scale.

The factors are represented on the first two lines of the

accompanying figure, and the work is carried on exactly as in our common system.

NONARY.

		7 1	8 4	5 2	3 5	7			
	7 3	8 4 1	5 7 6 4	3 3 8 3	7 6 1 7	1 7 0	5 0	8	
1	2	6	6	1	3	8	5	8	

EXAMPLES.

On the senary scale multiply 51,2041 by 34,4105, and express the result on the denary scale.

On the nonary scale multiply 7,6237 by 1,8065 and by ,05814, and express the product in decimals.

As the duodecimal system is still in use for the measurement of timber and stone, it may be worth while to examine it somewhat more closely.

If we have a board 5 feet 7 inches 11 lines long, by 2 feet 3 inches 7 lines broad, and wish to compute the number of square feet in its surface, we multiply the number 5...7...11 by 2...3...7 on the duodecimal scale: the operation stands thus,

```
   5,...7...11
   2,...3... 7
 ----------------------
  11,...3...10
   1,...4...11...9
         3... 3...7...5
 ----------------------
  13,...0... 1...4...5
```

The integer part of the result represents 13 square feet: the number in the next place represents twelfths of square feet, that is, rectangles one foot long and one inch broad; of these there

happens to be none in this example. The number in the second fraction place represents one square inch; then we have four-twelfths of a square inch and five square lines; and the whole may be read, 13 square feet, 1 square inch, and 53 square lines.

If, again, we wish to compute the solidity of a block having the above for its length and breadth, and of which the height is 4 feet 9 inches 5 lines, we proceed to multiply the above product by the number 4,...9...5, as under.

```
13,... 0... 1... 4... 5
 4,... 9... 5
52,... 0... 5... 5... 8
 9,... 9... 1... 0... 3... 9
       5... 5... 0... 6...10... 1
---------------------------------
62,... 2...11... 6... 6... 7... 1
```

which shows the result 62 cubic feet, 2 twelfths of a cubic foot (that is, blocks one foot square and one inch thick), 11 one-hundred-and-forty-fourths of a cubic foot (that is, rods one foot long and one inch square), 6 cubic inches, and so on. It is to be carefully noted that the third fractional place is that which represents cubic inches, and the sixth that for cubic lines. The result may be put under the form, 62 cubic feet, 426 cubic inches, and 949 cubic lines.

It is much more convenient to calculate in this way, than to reduce all to lines, multiply, and then reconvert into cubic feet. Here we have a clear illustration of the advantage of following a uniform system in the subdivision of measures.

When any of the dimensions exceed 12 feet, it is best to represent them on the duodecimal scale; thus for 172 feet 3 inches 7 lines, we would substitute 1...2...4,...3...7 when manipulating.

Exercises.

Required the surface of a court 227 feet 11 inches 3 lines long, by 172 feet 3 inches 7 lines broad.

Required the surface of a floor 32 feet 9 inches 8 lines long, by 22 feet 7 inches 11 lines broad.

What is the bulk of a block 83 feet 7 inches 10 lines long, 13 feet 11 inches 7 lines broad, and 8 feet 9 inches 4 lines thick ?

Required the square on a line 7 feet 3 inches 9 lines long.

Required the solidity of a cube, of which the side is 11 feet 7 inches 2 lines.

When the surface and one side of a rectangle are known. we obtain the other side by dividing the number which represents the area by that which represents the known side.

Thus if we be asked, " What must be the length of a board 1 foot 2 inches 5 lines broad, in order that its surface may contain 28 square feet, 91 square inches, and 19 square lines ; we represent the last named quantity on the duodecimal scale as

2...4‚...7...7...1...7,

and divide this number by 1‚...2...5 ; the operation being as under,

$$
\begin{array}{r|l|l}
1_{\text{‚}}\ldots2\ldots5 & 2\ldots4_{\text{‚}}\ldots\ 7\ldots7\ldots1\ldots7 & 1\ldots11_{\text{‚}}\ldots9\ldots11\ldots11\ldots7 \\
 & \underline{1\ldots2_{\text{‚}}\ldots\ 5} & \\
 & 1\ldots2_{\text{‚}}\ldots\ 2\ldots7 & \\
 & \underline{1\ldots1_{\text{‚}}\ldots\ 2\ldots7} & \\
 & \quad 1_{\text{‚}}\ldots\ 0\ldots0\ldots1 & \\
 & \quad \underline{{}_{\text{‚}}\ldots10\ldots9\ldots9} & \\
 & \qquad 1\ldots2\ldots4\ldots7 & \\
 & \qquad \underline{1\ldots1\ldots2\ldots7} & \\
 & \qquad\quad 1\ldots2\ldots0\ldots0 & \\
 & \qquad\quad \underline{1\ldots1\ldots2\ldots7} & \\
 & \qquad\qquad\quad 9\ldots5 &
\end{array}
$$

whence the required length is 23 feet, 9 inches, 11 lines, 11 twelfths of a line, and 7 one-hundred-and-forty-fourths.

EXERCISES.

A board is 1 foot 5 inches three lines broad ; what length must be cut off it to give a surface of 15 square feet, 117 square inches, and 89 square lines ?

A block of stone is 4 feet 6 inches 7 lines long, 2 feet 11 inches 7 lines broad; what must be its thickness, in order that it may contain 17 cubic feet, 179 cubic inches, and 1063 cubic lines?

The extraction of square roots may also be carried on in duodecimals, and though of no great practical value, now that the decimal notation is so generally used, the practice of it may serve to give maturity to the ideas of the student.

Let it be proposed, for example, to compute the side of a square which may contain 523 square feet, 106 square inches, and 97 square lines.

This on the duodecimal scale becomes 3...7...7,...8...10...8...1 square feet. The operation, exactly analogous to that in decimal arithmetic, is,

1	3...7... 7,...8...10...8...1	1...10,...10...7...5...11...4
1	1	
2...10	2...7... 7	
10	2...4... 4	
3... 8...10	3... 3,...8...10	
10	3 .. 1,...4... 4	
3... 9... 8...7	2,...4... 6...8... 1	
7	2,...2... 8...0... 1	
3... 9... 9...2... 5	1...10...8... 0	
...5	1... 7...0...10	
3... 9... 9...2...10...11	3...7... 2	
11	3...6... 0	
	1... 2	
	1... 3	

from which the side of the square is found to be 22 feet, 10 inches, 7 lines, &c.

EXERCISES.

Required the side of a square which may contain 79 square feet, 111 square inches, and 73 square lines.

Required the square root of 107 in duodecimals.

It is unnecessary to exemplify the processes of subtraction, division, etc., on other scales, since their principles and arrangement are identic with those which we have already studied on the denary scale. So, having now seen how to represent numbers and fractions, and how to perform the elementary operations upon any numeration scale, we may proceed to consider the leading properties of the scales. Some of these properties are general, that is, are common to all scales; others are special, belonging to individual scales, or to particular classes of them.

The first property of numeration scales, and one which has a most intimate connection with the Theory of Equations, is this, that

On every scale a counter placed in any groove is of more value than all the counters required to fill up the inferior grooves.

This flows from the very nature of the notation. If we fill up several grooves, beginning at the units' place, with as many counters as the scale allows of, and if we add unit to the number thus represented, the number of the counters in the units' place, being now equal to the basis of the scale, these counters may be removed, and a single counter placed for them in the second groove; and this process may go on until we have a single counter in the groove immediately above those which were filled, so that this single counter represents a number greater than that indicated by all the counters in the lower grooves. Thus if to the number

NONARY.

8 8 8 8 8 8 8

unit be added, the 9 in the units' place becomes a zero, and 1 is carried to the next place; this makes 9 in the second place or zero there, and unit in the next place. The result then is

NONARY.

1 0 0 0 0 0 0 0,

so that unit in the 8th place is greater than the greatest possible number represented on seven places.

This statement may be extended to the fractional part of the scale, and we may say that a counter in any place is equivalent to the counters which fill all the inferior places carried out without end. Thus the repeating fraction

DENARY.

.9 9 9 9 9 9 9, &c.,

is just equal to unit. For if we stop at the term .9, this wants one-tenth of being unit; if we take two terms, .99 wants one-hundredth of being unit, if we take three terms, .999 wants only one-thousandth of being unit; and thus the deficiency is reduced ten times with every additional term, so that if the series were carried out very far the defect from unit would be excessively minute.

In the same way the repeating fraction

QUINARY.

.4 4 4 4 4 4 4 4, &c.,

when conceived to be carried to an unlimited length, is just equivalent to unit; for the defect of .4 is .1, or one-fifth; the defect of .44 is .01, or one twenty-fifth; that of .4444 is .0001, or one six-hundred-and-twenty-fifth, and so on, the defect becoming five times smaller with each additional term.

Hence it seems that unit may be regarded as the sum of any of the interminate series;

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \&c.$$
$$\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \frac{2}{243} + \&c.$$
$$\frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \frac{3}{256} + \frac{3}{1024} + \&c.$$
$$\frac{10}{11} + \frac{10}{121} + \frac{10}{1331} + \frac{10}{14641} + \&c.$$

and so on.

We found that when a number is divisible by 9, the sum of the digits which express it on the denary scale is divisible by 9. The same statement may be made in regard to any other scale.

If a number be divisible by the number immediately less than the basis of the scale on which it is represented, the sum of its digits is also divisible by that number.

Thus if a number be divisible by 8, the sum of the digits

which represent it on the nonary scale is divisible by 8. The demonstration of this is merely a paraphrase of that which has been already given (Chap. VI.) in regard to the denary scale.

If, on the nonary scale, we add 8 to any number, we may augment the number of counters by 8 as when the units' place is empty; we may not alter the number of counters as when there are already counters in the units' place, and fewer than 8 counters in the second place, for it is enough to take a counter from the units and place it among the nines; or we may diminish the number of counters by 8, as when there are already counters in the units' groove, 8 counters in the nines' groove, and fewer than 8 counters in the groove for eighty-ones. Thus, generally, the addition of 8 to any number does not change the divisibility of the number of the counters by 8; and the same may be said of subtraction. Hence if a number represented on the nonary abacus contain eights with a remainder, the number of its counters must contain eights with the same remainder.

Similar reasoning may be applied to any other numeration scale, and thus we see that the well-known test for divisibility by nine is a mere case of a general law.

On the denary scale the test for divisibility by nine also answers for three; that is to say, if the sum of the digits of a number be divisible by three, the number itself is divisible by three: and this is because 3 is a divisor of 9.

In the same way, on the nonary scale, if the sum of the digits of a number be divisible by 8, by 4, or by 2, the number itself is divisible; and this because 4 and 2 are divisors of 8.

On the duodenary scale, eleven possesses this property; now eleven is a prime number, and therefore it is the only one for which this test can be used.

It was shown (Chap. VI.) that, of any number divisble by eleven, the sum of the one set of alternate digits, on the denary scale, is either equal to the sum of the other set, or differs from it by a number of elevens. A similar property belongs to the number immediately above the basis of any other scale. Thus

on the duodenary abacus, the number thirteen possesses this property.

Let us examine the matter closely. Thirteen is represented on the duodenary abacus by two counters, one on each of the two first grooves, or in writing by 1...1. Twice thirteen must then be written 2...2; thrice thirteen 3...3, and so on, up to eleven times thirteen, which is 11...11. Eleven times thirteen then wants just unit of being twelve times twelve.

The same is true of any other scale, that is, the product of the number immediately above by the number immediately below the basis is less than the square of the basis by unit, and this is in accordance with the law mentioned in p. 6.

It follows from this that if we take a counter from the units' place and put it in the third place, or move a counter two steps either to the right or left on the duodenary abacus, we do not alter the divisibility of the number by thirteen. Hence if all the counters in the first, third, fifth, etc., grooves, were collected in the first groove, and all the counters in the second, fourth, sixth, etc. grooves were collected in the second groove, the divisibility by thirteen would not be altered. But by throwing out simultaneously a counter from each of these two grooves, we subtract thirteen; and if we continue this until the counters in one of the grooves be exhausted, those in the other must also be exhausted, or there must remain a number divisible by thirteen, if the original number were so divisible.

The very same reasoning may be applied to any other scale.

The number eleven is prime, and thus it is the only number for which this test of divisibility can be used on the denary scale. But when we have to do with the nonary scale, ten and its two divisors 5 and 2 possess this character; that if the difference between the sums of the sets of alternate digits be 0, or be divisible by ten, five, or two, the number itself is so divisible.

The student may exercise himself in ascertaining to what cases this criterion of divisibility may be applied when the various scales are used.

When we attempt to represent a fraction on any numeration scale, we find that, unless the factors of the denominator be all factors of the basis of the scale, the series is interminate, and that the digits eventually circulate. If a fraction, supposed to be in its lowest terms, be represented on any scale, the number of the digits in the circulator can never be equal to the denominator of the fraction. Thus if we represent the fraction $\frac{1}{7}$ on any scale, the circulator can never have more than 6 places; for the residue at a division can only be one of the numbers 1, 2, 3, 4, 5, 6, and therefore at the seventh division one of the preceding remainders must occur, and must be followed by the other remainders in the same order: thus—

Binary, $\frac{1}{7}$	=	‚0 0 1 0 0 1 0 0 1 0 0 1, &c.
Ternary,	=	‚0 1 0 2 1 2 0 1 0 2 1 2, &c.
Quaternary,	=	‚0 2 1 0 2 1 0 2 1 0 2 1, &c.
Quinary,	=	‚0 3 2 4 1 2 0 3 2 4 1 2, &c.
Senary,	=	‚0 5 0 5 0 5 0 5 0 5 0 5, &c.
		&c. &c.

In all scales the fraction which has the number immediately less than the basis of the scale for its denominator and unit for its numerator is represented by unit repeated in the fractional places of the scale: thus—

Binary, $\frac{1}{1}$	=	‚1 1 1 1 1 1 1, &c.
Ternary, $\frac{1}{2}$	=	‚1 1 1 1 1 1 1, &c.
Quaternary, $\frac{1}{3}$	=	‚1 1 1 1 1 1 1, &c.

and so on; or, in our usual notation,

$$\frac{1}{1} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \text{\&c.}$$
$$\frac{1}{2} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \text{\&c.}$$
$$\frac{1}{3} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \text{\&c.}$$
$$\frac{1}{4} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625} + \text{\&c.}$$

and this must be so since, on performing the division, there is unit over at each step.

Hence the only divisors which can give single repeaters on any scale are the number immediately below the basis of the

scale and the divisors of that number. Thus on the denary scale the only divisors which give simple repeaters are 9 and 3; on the septenary scale, 6, 3, and 2; on the nonary scale, 8, 4, 2; while on the duodenary scale the only divisor which gives a simple repetent is 11.

In the very same way it can be shown that fractions circulating in periods of two terms are produced by the divisor one less than the square of the basis, or by the aliquot parts of that divisor, among which the number immediately above the basis is always to be found. Thus, on the nonary scale, circulators of two terms arise from divisions by 80, or by any of the divisors of 80; so that, excluding those of which both terms are alike, and which are simple repeaters, the divisors producing periods of two terms on the nonary scale are *eighty*, *forty*, *twenty*, *ten*, and *five*.

The student may find it an agreeable exercise to investigate the laws of circulation in three, four, etc. terms, for other scales, as has been done for the denary scale in Chap. VI.

The mode of counting in tens has found favour among all nations. Whatever may be the reason for this preference, it seems clear that no consideration of the comparative advantages of the decimal system had any influence in procuring its adoption, since, at the time of its origin, nothing was known of the peculiar properties of it or of any other system of numeration.

It has been argued, and with great plausibility, that it would be much more convenient to count in *dozens*, *grosses*, and *double-grosses*. The reason is assigned, that the number twelve has more divisors than the number ten, and that, therefore, there would be more convenience in the working of fractions. The duodecimal system is actually in use, and artificers who measure in feet, inches, and lines, naturally employ the duodenary scale in their computations.

Now we are hardly in a position to decide impartially in the matter, seeing that we have been used from infancy to count in

tens. We must endeavour to get rid of the prejudice of habit, and try to imagine what would have been the facilities of the duodecimal scale if our language had been suitable, and if we had been brought up to use it.

Addition and subtraction are performed on this just as on any other scale, and are very slightly if at all affected by its peculiarities. It is in multiplication and division that the effects of these peculiarities are to be looked for.

Doubling is easy on the duodecimal scale; whenever a term is above 6 unit must be carried to the higher place of the double: thus—

2... 7... 4,... 1... 8... 3... 5
5... 2... 8,... 3... 4... 6...10
10... 5... 4,... 6... 9... 1... 8
1... 8...10... 9,... 1... 6... 3... 4

and so on;

from which it is apparent that multiplication by 2 is as easily performed on this scale as on the decimal one.

Tripling is also quite easy, since 4 in any place sends up 1, and 8 sends up 2; as may be seen in this example:

2... 7... 4,... 1... 8... 3... 5
7...10... 0,... 5... 0...10... 3
1...11... 6... 1,... 3... 2... 6... 9
5...10... 6... 3,... 9... 7... 8... 3

Tripling, then, is fully easier on this scale than on the common one; and it is to be noted that all numbers which are divisible by 3 end in 3, 6, 9, or 0.

When multiplied by 4, *three* sends up 1, *six* sends up 2, and *nine* sends up 3: thus—

2... 7... 4,... 1... 8... 3... 5
10... 5... 4,... 6... 9... 1... 8
3... 5... 9... 6,... 3... 0... 6... 8

It is very evident from these examples that, if our language had been made to suit the duodecimal system, multiplication by

3 and by 4, would have been easier than on the decimal system, and that the multiples by 3, 4, 8, and 9, would have been readily fixed in the memory; just as the multiples of 5 are easily recollected now.

The next multiplier 5 presents on the duodecimal scale characters similar to those which 7 exhibits on the decimal scale. The fractions $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, and $\frac{4}{5}$ are

,2...4...9...7...2...4...9...7 &c.
,4...9...7...2...4...9...7...2 &c.
,7...2...4...9...7...2...4...9 &c.
,9...7...2...4...9...7...2...4 &c.

and these mark the limits at which 1, 2, 3, or 4 are sent up. Multiplication by 5, then, is nearly as troublesome as that by 7 is on our present system.

Multiplication by 6 in duodecimals is quite analogous to that by 5 in decimals.

The fractions having seven for their denominator become,

, 1... 8... 6...10... 3... 5... 1... 8... 6...10... 3... 5 &c.
, 3... 5... 1... 8... 6...10... 3... 5... 1... 8... 6...10 &c.
, 5... 1... 8... 6...10... 3... 5... 1... 8... 6...10... 3 &c.
, 6...10... 3... 5... 1... 8... 6...10... 3... 5... 1... 8 &c.
, 8... 6...10... 3... 5... 1... 8... 6...10... 3... 5... 1 &c.
,10... 3... 5... 1... 8... 6...10... 3... 5... 1... 8... 6 &c.

and thus multiplication by 7 becomes fully more troublesome on this scale than on the common one.

The fractions having 8 for their denominators are represented by duodecimal fractions of two places, viz.

$\frac{1}{8}$ = ,1...6	$\frac{3}{8}$ = ,4...6	$\frac{5}{8}$ = ,7...6	$\frac{7}{8}$ = ,10...6
$\frac{2}{8}$ = ,3...0	$\frac{4}{8}$ = ,6...0	$\frac{6}{8}$ = ,9...0	

so that multiplication by 8 is easy on this scale.

The same may be said of multiplication by 9; for,

$\frac{1}{9}$ = ,1...4	$\frac{3}{9}$ = ,4...0	$\frac{5}{9}$ = ,6...8	$\frac{7}{9}$ = 9...4
$\frac{2}{9}$ = ,2...8	$\frac{4}{9}$ = ,5...4	$\frac{6}{9}$ = ,8...0	$\frac{8}{9}$ = 10...8

Fractions with the denominator 10 are interminate, the cir-

culator being the same as that for 5, and the odd tenths having one non-circulating figure; thus,

$\frac{1}{10}=$	,1...2...4...9...7...2...4...9...7	&c.
$\frac{2}{10}=$	,2...4...9...7...2...4...9...7...2	&c.
$\frac{3}{10}=$	,3...7...2...4...9...7...2...4...9	&c.
$\frac{4}{10}=$	,4...9...7...2...4...9...7...2...4	&c.
$\frac{5}{10}=$	,6...0...0...0...0...0...0...0...0	
$\frac{6}{10}=$	,7...2...4...9...7...2...4...9...7	&c.
$\frac{7}{10}=$	,8...4...9...7...2...4...9...7...2	&c.
$\frac{8}{10}=$	,9...7...2...4...9...7...2...4...9	&c.
$\frac{9}{10}=$	,10...9...7...2...4...9...7...2...4	&c.

So that multiplication by 10 is nearly as difficult as that by 7 on the common scale.

Multiplication by 11 exactly resembles our ordinary multiplication by 9, the fractions *elevenths* being all simple repeaters.

It thus seems that the management of the duodecimal scale would present a greater variety of cases, and be fully more intricate, than the decimal system, and it is very much to be questioned whether the having *thirds* instead of *fifths*, expressible in finite terms, would form any countervailing advantage.

If we look only to facility in manipulation, the preference must be given to the senary scale, because the factors of all the numbers below 6, excepting 5, are also factors of 6; so that all the operations of multiplication are easily performed. The progression of the senary scale is slow, and on the whole, we may admit that the choice of the denary scale has been most fortunate, since, without needing the recollection of an unmanageable set of products, and without involving any great intricacy in work, it yet progresses with a rapidity sufficient to satisfy all the requirements of business or of science.

APPENDIX.

TABLE OF QUARTER SQUARES.

0	0	50	625	100	2500	150	5625
1	0	51	650	01	2550	51	5700
2	1	52	676	02	2601	52	5776
3	2	53	702	03	2652	53	5852
4	4	54	729	04	2704	54	5929
5	6	55	756	05	2756	55	6006
6	9	56	784	06	2809	56	6084
7	12	57	812	07	2862	57	6162
8	16	58	841	08	2916	58	6241
9	20	59	870	09	2970	59	6320
10	25	60	900	110	3025	160	6400
11	30	61	930	11	3080	61	6480
12	36	62	961	12	3136	62	6561
13	42	63	992	13	3192	63	6642
14	49	64	1024	14	3249	64	6724
15	56	65	1056	15	3306	65	6806
16	64	66	1089	16	3364	66	6889
17	72	67	1122	17	3422	67	6972
18	81	68	1156	18	3481	68	7056
19	90	69	1190	19	3540	69	7140
20	100	70	1225	120	3600	170	7225
21	110	71	1260	21	3660	71	7310
22	121	72	1296	22	3721	72	7396
23	132	73	1332	23	3782	73	7482
24	144	74	1369	24	3844	74	7569
25	156	75	1406	25	3906	75	7656
26	169	76	1444	26	3969	76	7744
27	182	77	1482	27	4032	77	7832
28	196	78	1521	28	4096	78	7921
29	210	79	1560	29	4160	79	8010
30	225	80	1600	130	4225	180	8100
31	240	81	1640	31	4290	81	8190
32	256	82	1681	32	4356	82	8281
33	272	83	1722	33	4422	83	8372
34	289	84	1764	34	4489	84	8464
35	306	85	1806	35	4556	85	8556
36	324	86	1849	36	4624	86	8649
37	342	87	1892	37	4692	87	8742
38	361	88	1936	38	4761	88	8836
39	380	89	1980	39	4830	89	8930
40	400	90	2025	140	4900	190	9025
41	420	91	2070	41	4970	91	9120
42	441	92	2116	42	5041	92	9216
43	462	93	2162	43	5112	93	9312
44	484	94	2209	44	5184	94	9409
45	506	95	2256	45	5256	95	9506
46	529	96	2304	46	5329	96	9604
47	552	97	2352	47	5402	97	9702
48	576	98	2401	48	5476	98	9801
49	600	99	2450	49	5550	99	9900

200	1 0000	250	1 5625	300	2 2500	350	3 0625
01	1 0100	51	1 5750	01	2 2650	51	3 0800
02	1 0201	52	1 5876	02	2 2801	52	3 0976
03	1 0302	53	1 6002	03	2 2952	53	3 1152
04	1 0404	54	1 6129	04	2 3104	54	3 1329
05	1 0506	55	1 6256	05	2 3256	55	3 1506
06	1 0609	56	1 6384	06	2 3409	56	3 1684
07	1 0712	57	1 6512	07	2 3562	57	3 1862
08	1 0816	58	1 6641	08	2 3716	58	3 2041
09	1 0920	59	1 6770	09	2 3870	59	3 2220
210	1 1025	260	1 6900	310	2 4025	360	3 2400
11	1 1130	61	1 7030	11	2 4180	61	3 2580
12	1 1236	62	1 7161	12	2 4336	62	3 2761
13	1 1342	63	1 7292	13	2 4492	63	3 2942
14	1 1449	64	1 7424	14	2 4649	64	3 3124
15	1 1556	65	1 7556	15	2 4806	65	3 3306
16	1 1664	66	1 7689	16	2 4964	66	3 3489
17	1 1772	67	1 7822	17	2 5122	67	3 3672
18	1 1881	68	1 7956	18	2 5281	68	3 3856
19	1 1990	69	1 8090	19	2 5440	69	3 4040
220	1 2100	270	1 8225	320	2 5600	370	3 4225
21	1 2210	71	1 8360	21	2 5760	71	3 4410
22	1 2321	72	1 8496	22	2 5921	72	3 4596
23	1 2432	73	1 8632	23	2 6082	73	3 4782
24	1 2544	74	1 8769	24	2 6244	74	3 4969
25	1 2656	75	1 8906	25	2 6406	75	3 5156
26	1 2769	76	1 9044	26	2 6569	76	3 5344
27	1 2882	77	1 9182	27	2 6732	77	3 5532
28	1 2996	78	1 9321	28	2 6896	78	3 5721
29	1 3110	79	1 9460	29	2 7060	79	3 5910
230	1 3225	280	1 9600	330	2 7225	380	3 6100
31	1 3340	81	1 9740	31	2 7390	81	3 6290
32	1 3456	82	1 9881	32	2 7556	82	3 6481
33	1 3572	83	2 0022	33	2 7722	83	3 6672
34	1 3689	84	2 0164	34	2 7889	84	3 6864
35	1 3806	85	2 0306	35	2 8056	85	3 7056
36	1 3924	86	2 0449	36	2 8224	86	3 7249
37	1 4042	87	2 0592	37	2 8392	87	3 7442
38	1 4161	88	2 0736	38	2 8561	88	3 7636
39	1 4280	89	2 0880	39	2 8730	89	3 7830
240	1 4400	290	2 1025	340	2 8900	390	3 8025
41	1 4520	91	2 1170	41	2 9070	91	3 8220
42	1 4641	92	2 1316	42	2 9241	92	3 8416
43	1 4762	93	2 1462	43	2 9412	93	3 8612
44	1 4884	94	2 1609	44	2 9584	94	3 8809
45	1 5006	95	2 1756	45	2 9756	95	3 9006
46	1 5129	96	2 1904	46	2 9929	96	3 9204
47	1 5252	97	2 2052	47	3 0102	97	3 9402
48	1 5376	98	2 2201	48	3 0276	98	3 9601
49	1 5500	99	2 2350	49	3 0450	99	3 9800

400	4 0000	450	5 0625	500	6 2500	550	7 5625
01	4 0200	51	5 0850	01	6 2750	51	7 5900
02	4 0401	52	5 1076	02	6 3001	52	7 6176
03	4 0602	53	5 1302	03	6 3252	53	7 6452
04	4 0804	54	5 1529	04	6 3504	54	7 6729
05	4 1006	55	5 1756	05	6 3756	55	7 7006
06	4 1209	56	5 1984	06	6 4009	56	7 7284
07	4 1412	57	5 2212	07	6 4262	57	7 7562
08	4 1616	58	5 2441	08	6 4516	58	7 7841
09	4 1820	59	5 2670	09	6 4770	59	7 8120
410	4 2025	460	5 2900	510	6 5025	560	7 3400
11	4 2230	61	5 3130	11	6 5280	61	7 8680
12	4 2436	62	5 3361	12	6 5536	62	7 8961
13	4 2642	63	5 3592	13	6 5792	63	7 9242
14	4 2849	64	5 3824	14	6 6049	64	7 9524
15	4 3056	65	5 4056	15	6 6306	65	7 9806
16	4 3264	66	5 4289	16	6 6564	66	8 0089
17	4 3472	67	5 4522	17	6 6822	67	8 0372
18	4 3681	68	5 4756	18	6 7081	68	8 0656
19	4 3890	69	5 4990	19	6 7340	69	8 0940
420	4 4100	470	5 5225	520	6 7600	570	8 1225
21	4 4310	71	5 5460	21	6 7860	71	8 1510
22	4 4521	72	5 5696	22	6 8121	72	8 1796
23	4 4732	73	5 5932	23	6 8382	73	8 2082
24	4 4944	74	5 6169	24	6 8644	74	8 2369
25	4 5156	75	5 6406	25	6 8906	75	8 2656
26	4 5369	76	5 6644	26	6 9169	76	8 2944
27	4 5582	77	5 6882	27	6 9432	77	8 3232
28	4 5796	78	5 7121	28	6 9696	78	8 3521
29	4 6010	79	5 7360	29	6 9960	79	8 3810
430	4 6225	480	5 7600	530	7 0225	580	8 4100
31	4 6440	81	5 7840	31	7 0490	81	8 4390
32	4 6656	82	5 8081	32	7 0756	82	8 4681
33	4 6872	83	5 8322	33	7 1022	83	8 4972
34	4 7089	84	5 8564	34	7 1289	84	8 5264
35	4 7306	85	5 8806	35	7 1556	85	8 5556
36	4 7524	86	5 9049	36	7 1824	86	8 5849
37	4 7742	87	5 9292	37	7 2092	87	8 6142
38	4 7961	88	5 9536	38	7 2361	88	8 6436
39	4 8180	89	5 9780	39	7 2630	89	8 6730
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41	4 8620	91	6 0270	41	7 3170	91	8 7320
42	4 8841	92	6 0516	42	7 3441	92	8 7616
43	4 9062	93	6 0762	43	7 3712	93	8 7912
44	4 9284	94	6 1009	44	7 3984	94	8 8209
45	4 9506	95	6 1256	45	7 4256	95	8 8506
46	4 9729	96	6 1504	46	7 4529	96	8 8804
47	4 9952	97	6 1752	47	7 4802	97	8 9102
48	5 0176	98	6 2001	48	7 5076	98	8 9401
49	5 0400	99	6 2250	49	7 5350	99	8 9700

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03	9 0902	53	10 6602	03	12 3552	53	14 1752
04	9 1204	54	10 6929	04	12 3904	54	14 2129
05	9 1506	55	10 7256	05	12 4256	55	14 2506
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07	9 2112	57	10 7912	07	12 4962	57	14 3262
08	9 2416	58	10 8241	08	12 5316	58	14 3641
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12	9 3636	62	10 9561	12	12 6736	62	14 5161
13	9 3942	63	10 9892	13	12 7092	63	14 5542
14	9 4249	64	11 0224	14	12 7449	64	14 5924
15	9 4556	65	11 0556	15	12 7806	65	14 6306
16	9 4864	66	11 0889	16	12 8164	66	14 6689
17	9 5172	67	11 1222	17	12 8522	67	14 7072
18	9 5481	68	11 1556	18	12 8881	68	14 7456
19	9 5790	69	11 1890	19	12 9240	69	14 7840
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22	9 6721	72	11 2896	22	13 0321	72	14 8996
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24	9 7344	74	11 3569	24	13 1044	74	14 9769
25	9 7656	75	11 3906	25	13 1406	75	15 0156
26	9 7969	76	11 4244	26	13 1769	76	15 0544
27	9 8282	77	11 4582	27	13 2132	77	15 0932
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29	9 8910	79	11 5260	29	13 2860	79	15 1710
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36	10 1124	86	11 7649	36	13 5424	86	15 4449
37	10 1442	87	11 7992	37	13 5792	87	15 4842
38	10 1761	88	11 8336	38	13 6161	88	15 5236
39	10 2080	89	11 8680	39	13 6530	89	15 5630
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41	10 2720	91	11 9370	41	13 7270	91	15 6420
42	10 3041	92	11 9716	42	13 7641	92	15 6816
43	10 3362	93	12 0062	43	13 8012	93	15 7212
44	10 3684	94	12 0409	44	13 8384	94	15 7609
45	10 4006	95	12 0756	45	13 8756	95	15 8006
46	10 4329	96	12 1104	46	13 9129	96	15 8404
47	10 4652	97	12 1452	47	13 9502	97	15 8802
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03	16 1202	53	18 1902	03	20 3852	53	22 7052
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16	16 6464	66	18 7489	16	20 9764	66	23 3289
17	16 6872	67	18 7922	17	21 0222	67	23 3772
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19	16 7690	69	18 8790	19	21 1140	69	23 4740
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21	16 8510	71	18 9660	21	21 2060	71	23 5710
22	16 8921	72	19 0096	22	21 2521	72	23 6196
23	16 9332	73	19 0532	23	21 2982	73	23 6682
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26	17 0569	76	19 1844	26	21 4369	76	23 8144
27	17 0982	77	19 2282	27	21 4832	77	23 8632
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29	17 1810	79	19 3160	29	21 5760	79	23 9610
830	17 2225	880	19 3600	930	21 6225	980	24 0100
31	17 2640	81	19 4040	31	21 6690	81	24 0590
32	17 3056	82	19 4481	32	21 7156	82	24 1081
33	17 3472	83	19 4922	33	21 7622	83	24 1572
34	17 3889	84	19 5364	34	21 8089	84	24 2064
35	17 4306	85	19 5806	35	21 8556	85	24 2556
36	17 4724	86	19 6249	36	21 9024	86	24 3049
37	17 5142	87	19 6692	37	21 9492	87	24 3542
38	17 5561	88	19 7136	38	21 9961	88	24 4036
39	17 5980	89	19 7580	39	22 0430	89	24 4530
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41	17 6820	91	19 8470	41	22 1370	91	24 5520
42	17 7241	92	19 8916	42	22 1841	92	24 6016
43	17 7662	93	19 9362	43	22 2312	93	24 6512
44	17 8084	94	19 9809	44	22 2784	94	24 7009
45	17 8506	95	20 0256	45	22 3256	95	24 7506
46	17 8929	96	20 0704	46	22 3729	96	24 8004
47	17 9352	97	20 1152	47	22 4202	97	24 8502
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03	25 1502	53	27 7202	03	30 4152	53	33 2352
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06	25 3009	56	27 8784	06	30 5809	56	33 4084
07	25 3512	57	27 9312	07	30 6362	57	33 4662
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12	25 6036	62	28 1961	12	30 9136	62	33 7561
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14	25 7049	64	28 3024	14	31 0249	64	33 8724
15	25 7556	65	28 3556	15	31 0806	65	33 9306
16	25 8064	66	28 4089	16	31 1364	66	33 9889
17	25 8572	67	28 4622	17	31 1922	67	34 0472
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19	25 9590	69	28 5690	19	31 3040	69	34 1640
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22	26 1121	72	28 7296	22	31 4721	72	34 3396
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24	26 2144	74	28 8369	24	31 5844	74	34 4569
25	26 2656	75	28 8906	25	31 6406	75	34 5156
26	26 3169	76	28 9444	26	31 6969	76	34 5744
27	26 3682	77	28 9982	27	31 7532	77	34 6332
28	26 4196	78	29 0521	28	31 8096	78	34 6921
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32	26 6256	82	29 2681	32	32 0356	82	34 9281
33	26 6772	83	29 3222	33	32 0922	83	34 9872
34	26 7289	84	29 3764	34	32 1489	84	35 0464
35	26 7806	85	29 4306	35	32 2056	85	35 1056
36	26 8324	86	29 4849	36	32 2624	86	35 1649
37	26 8842	87	29 5392	37	32 3192	87	35 2242
38	26 9361	88	29 5936	38	32 3761	88	35 2836
39	26 9880	89	29 6480	39	32 4330	89	35 3430
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42	27 1441	92	29 8116	42	32 6041	92	35 5216
43	27 1962	93	29 8662	43	32 6612	93	35 5812
44	27 2484	94	29 9209	44	32 7184	94	35 6409
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46	27 3529	96	30 0304	46	32 8329	96	35 7604
47	27 4052	97	30 0852	47	32 8902	97	35 8202
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06	36 3609	56	39 4384	06	42 6409	56	45 9684
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15	36 9056	65	40 0056	15	43 2306	65	46 5806
16	36 9664	66	40 0689	16	43 2964	66	46 6489
17	37 0272	67	40 1322	17	43 3622	67	46 7172
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22	37 3321	72	40 4496	22	43 6921	72	47 0596
23	37 3932	73	40 5132	23	43 7582	73	47 1282
24	37 4544	74	40 5769	24	43 8244	74	47 1969
25	37 5156	75	40 6406	25	43 8906	75	47 2656
26	37 5769	76	40 7044	26	43 9569	76	47 3344
27	37 6382	77	40 7682	27	44 0232	77	47 4032
28	37 6996	78	40 8321	28	44 0896	78	47 4721
29	37 7610	79	40 8960	29	44 1560	79	47 5410
1230	37 8225	1280	40 9600	1330	44 2225	1380	47 6100
31	37 8840	81	41 0240	31	44 2890	81	47 6790
32	37 9456	82	41 0881	32	44 3556	82	47 7481
33	38 0072	83	41 1522	33	44 4222	83	47 8172
34	38 0689	84	41 2164	34	44 4889	84	47 8864
35	38 1306	85	41 2806	35	44 5556	85	47 9556
36	38 1924	86	41 3449	36	44 6224	86	48 0249
37	38 2542	87	41 4092	37	44 6892	87	48 0942
38	38 3161	88	41 4736	38	44 7561	88	48 1636
39	38 3780	89	41 5380	39	44 8230	89	48 2330
1240	38 4400	1290	41 6025	1340	44 8900	1390	48 3025
41	38 5020	91	41 6670	41	44 9570	91	48 3720
42	38 5641	92	41 7316	42	45 0241	92	48 4416
43	38 6262	93	41 7962	43	45 0912	93	48 5112
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17	50 1972	67	53 8022	17	57 5322	67	61 3872
18	50 2681	68	53 8756	18	57 6081	68	61 4656
19	50 3390	69	53 9490	19	57 6840	69	61 5440
1420	50 4100	1470	54 0225	1520	57 7600	1570	61 6225
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22	50 5521	72	54 1696	22	57 9121	72	61 7796
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24	50 6944	74	54 3169	24	58 0644	74	61 9369
25	50 7656	75	54 3906	25	58 1406	75	62 0156
26	50 8369	76	54 4644	26	58 2169	76	62 0944
27	50 9082	77	54 5382	27	58 2932	77	62 1732
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32	51 2656	82	54 9081	32	58 6756	82	62 5681
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35	51 4806	85	55 1306	35	58 9056	85	62 8056
36	51 5524	86	55 2049	36	58 9824	86	62 8849
37	51 6242	87	55 2792	37	59 0592	87	62 9642
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39	51 7680	89	55 4280	39	59 2130	89	63 1230
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42	51 9841	92	55 6516	42	59 4441	92	63 3616
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46	52 2729	96	55 9504	46	59 7529	96	63 6804
47	52 3452	97	56 0252	47	59 8302	97	63 7602
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03	64 2402	53	68 3102	03	72 5052	53	76 8252
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15	65 2056	65	69 3056	15	73 5306	65	77 8806
16	65 2864	66	69 3889	16	73 6164	66	77 9689
17	65 3672	67	69 4722	17	73 7022	67	78 0572
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24	65 9344	74	70 0569	24	74 3044	74	78 6769
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26	66 0969	76	70 2244	26	74 4769	76	78 8544
27	66 1782	77	70 3082	27	74 5632	77	78 9432
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31	66 5040	81	70 6440	31	74 9090	81	79 2990
32	66 5856	82	70 7281	32	74 9956	82	79 3881
33	66 6672	83	70 8122	33	75 0822	83	79 4772
34	66 7489	84	70 8964	34	75 1689	84	79 5664
35	66 8306	85	70 9806	35	75 2556	85	79 6556
36	66 9124	86	71 0649	36	75 3424	86	79 7449
37	66 9942	87	71 1492	37	75 4292	87	79 8342
38	67 0761	88	71 2336	38	75 5161	88	79 9236
39	67 1580	89	71 3180	39	75 6030	89	80 0130
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42	67 4041	92	71 5716	42	75 8641	92	80 2816
43	67 4862	93	71 6562	43	75 9512	93	80 3712
44	67 5684	94	71 7409	44	76 0384	94	80 4609
45	67 6506	95	71 8256	45	76 1256	95	80 5506
46	67 7329	96	71 9104	46	76 2129	96	80 6404
47	67 8152	97	71 9952	47	76 3002	97	80 7302
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06	81 5409	56	86 1184	06	90 8209	56	95 6484
07	81 6312	57	86 2112	07	90 9162	57	95 7462
08	81 7216	58	86 3041	08	91 0116	58	95 8441
09	81 8120	59	86 3970	09	91 1070	59	95 9420
1810	81 9025	1860	86 4900	1910	91 2025	1960	96 0400
11	81 9930	61	86 5830	11	91 2980	61	96 1380
12	82 0836	62	86 6761	12	91 3936	62	96 2361
13	82 1742	63	86 7692	13	91 4892	63	96 3342
14	82 2649	64	86 8624	14	91 5849	64	96 4324
15	82 3556	65	86 9556	15	91 6806	65	96 5306
16	82 4464	66	87 0489	16	91 7764	66	96 6289
17	82 5372	67	87 1422	17	91 8722	67	96 7272
18	82 6281	68	87 2356	18	91 9681	68	96 8256
19	82 7190	69	87 3290	19	92 0640	69	96 9240
1820	82 8100	1870	87 4225	1920	92 1600	1970	97 0225
21	82 9010	71	87 5160	21	92 2560	71	97 1210
22	82 9921	72	87 6096	22	92 3521	72	97 2196
23	83 0832	73	87 7032	23	92 4482	73	97 3182
24	83 1744	74	87 7969	24	92 5444	74	97 4169
25	83 2656	75	87 8906	25	92 6406	75	97 5156
26	83 3569	76	87 9844	26	92 7369	76	97 6144
27	83 4482	77	88 0782	27	92 8332	77	97 7132
28	83 5396	78	88 1721	28	92 9296	78	97 8121
29	83 6310	79	88 2660	29	93 0260	79	97 9110
1830	83 7225	1880	88 3600	1930	93 1225	1980	98 0100
31	83 8140	81	88 4540	31	93 2190	81	98 1090
32	83 9056	82	88 5481	32	93 3156	82	98 2081
33	83 9972	83	88 6422	33	93 4122	83	98 3072
34	84 0889	84	88 7364	34	93 5089	84	98 4064
35	84 1806	85	88 8306	35	93 6056	85	98 5056
36	84 2724	86	88 9249	36	93 7024	86	98 6049
37	84 3642	87	89 0192	37	93 7992	87	98 7042
38	84 4561	88	89 1136	38	93 8961	88	98 8036
39	84 5480	89	89 2080	39	93 9930	89	98 9030
1840	84 6400	1890	89 3025	1940	94 0900	1990	99 0025
41	84 7320	91	89 3970	41	94 1870	91	99 1020
42	84 8241	92	89 4916	42	94 2841	92	99 2016
43	84 9162	93	89 5862	43	94 3812	93	99 3012
44	85 0084	94	89 6809	44	94 4784	94	99 4009
45	85 1006	95	89 7756	45	94 5756	95	99 5006
46	85 1929	96	89 8704	46	94 6729	96	99 6004
47	85 2852	97	89 9652	47	94 7702	97	99 7002
48	85 3776	98	90 0601	48	94 8676	98	99 8001
49	85 4700	99	90 1550	49	94 9650	99	99 9000

ANSWERS TO EXAMPLES.

(*Page* 3.)

1 849	6 241	7 056
10 000	240 100	946 729
7 812 025	32 307 856	49 112 064
64 016 001	92 563 641	205 262 929

(*Page* 12.)

130 458	51 448	70 965
107 712	231 903	389 697
608 968	951 735	909 972
448 329	978 072	863 184

(*Page* 13.)

£364, 13s. 4·3d.

(*Page* 16.)

$\frac{4}{9}$	$\frac{81}{169}$	$\frac{576}{841}$	$\frac{8464}{491401}$	$\frac{1}{299929}$
$\frac{9}{25}$	$\frac{25}{361}$	$\frac{289}{1444}$	$\frac{11881}{89401}$	$\frac{552049}{929296}$
$\frac{25}{49}$	$\frac{1}{100}$	$\frac{1681}{2601}$	$\frac{2704}{219961}$	

$10\frac{39}{49}$	$26908\frac{9}{6724}$	$208\frac{9}{49}$	$424\frac{9}{25}$	$129430\frac{186}{289}$
$72\frac{217}{289}$	$104\frac{1}{25}$	$25546\frac{6582}{20449}$	$5695\frac{49}{225}$	$66768\frac{289}{1849}$

(*Page* 17.)

,251 001	128 022,986 809
,947 702 25	7 582,752 241
9,031 828 09	,010 114 727 184
,000 000 085 030 56	,001 497 69
1,001 924 925 444	9,037 719 438 4

(*Page* 19.)

27	474 552	335 702 375	22 330 474 496
343	1 000 000	567 663 552	59 638 983 643
3 375	2 571 353	731 432 701	164 837 013 587
17 576	7 880 599	3 892 119 517	994 908 665 087
166 375	19 902 511	5 773 874 184	

(*Page* 24.)

$\frac{1}{64}$	$\frac{729}{1000}$	$\frac{15625}{19683}$	$\frac{27}{39944812}$
$\frac{8}{27}$	$\frac{2744}{3375}$	$\frac{97336}{103823}$	$\frac{7077888}{7414875}$
$\frac{1}{125}$	$1\frac{721}{3375}$	$\frac{857375}{1030301}$	$\frac{184220009}{799178752}$
$\frac{27}{64}$	$\frac{4913}{9261}$	$\frac{912673}{2863288}$	$\frac{426957777}{895841344}$

$3\frac{3}{8}$	$993\ 348\frac{100}{729}$
$35\frac{162}{343}$	$163\ 368\frac{8}{27}$
$813\frac{1}{27}$	$8\ 935\ 304\frac{181880}{185193}$
$3\ 699\frac{3043}{3375}$	$35\ 561\ 354\frac{2137}{2197}$
$55\ 464\frac{9047}{10648}$	$653\ 440\ 347\frac{2197}{4913}$

(*Page* 25.)

42,875	,997 002 999
103,823	,985 074 875
7200,237 491	709 899,390 552 743
1367,631	2,226 435 085 860 552
1,092 727	125,014 250 541 506 859
1,003 003 001	,00 0000 000 926 859 375
1,015 075 125	8,045 325 330 982 633

(*Page* 26.)

81	78 074 896
625	104 060 401
4 096	639 128 961
10 000	333 621 760 000
28 561	426 231 402 496
104 976	988 053 892 081
531 441	1 004 006 004 001
6 765 201	1 063 772 210 101 441
21 381 376	6 404 947 515 799 056
57 289 761	12 531 684 988 464 096 016

$39\frac{1}{16}$ $1608\frac{73}{81}$ $\frac{1}{16}$ $\frac{81}{2401}$ $\frac{533794816}{547981281}$

$1\frac{4641}{10000}$ $132833\frac{2047}{14641}$ $\frac{1}{625}$ $\frac{1}{4879681}$ $1\frac{4060401}{100000000}$

$474\frac{22}{81}$ $\frac{625}{83521}$ $\frac{1}{10000}$ $\frac{10556001}{16777216}$

2,856 1 96 826,319 964 16

86,536 506 25 ,000 000 937 890 625

3976,498 143 989 361 400 517,584 516 726 397 337 6

,000 533 794 816 1,004 006 004 001

96 075 126,724 921 344 256

,000 000 000 000 104 821 185 121

23 372 086 087,589 538 991 402 867 506 25

116 472 876 292,942 043 022 360 985 6

(*Page* 34.)

1 024 161 051 $\frac{16807}{161051}$

16 807 $1\ 564\ 625\frac{157}{243}$ $262\ 064\frac{27105}{32768}$

$525\frac{7}{32}$ $\frac{243}{1024}$ $\frac{9509900499}{10000000000}$

15,053 664 563 2 152,753 985 827 571 424

1,020 160 641 281 024 ,000 657 748 550 151

,000 005 807 138 916 642 57

,000 000 000 000 048 117 014 085 7

(*Page* 38.)

6 103 515 625 ; 184 884 258 895 036 416 ;

$1\ 577\ 202\frac{57921073}{60466176}$; $\frac{1}{2187}$;

1,141 720 871 941 795 033 376 768 379 488 352 032 745 202 160 436 880 904 143 ;

,000 002 540 847 689 640 483 1 ;

,000 000 000 000 001 801 152 661 463 ;
,000 000 000 000 000 000 000 000 000 000 000 000 000 000 068 719 476 736 ;
124 554 829 371 809 625,521 988 622 383 405 844 834 489 639 063 493 3.

6^7 $(2\frac{1}{2})^{30}$ 73^{14} 13^6 17^{16} 5^{23}

(*Page* 39.)

3^{12} 7^{12} $(1\frac{3}{4})^2$ 13^5 43^{16} 19

2^{21} 5^{24} 17^{36} 23^{35} 23^{35} 13^{30} 29^{105}

(*Page* 40.)

$$\begin{matrix} 231^7 \times 77^4 \times 11^2 \\ 30^7 \times 6 \times 2^2 \end{matrix} \qquad \begin{matrix} 520^2 \times 40^3 \times 5^8 \\ 720^2 \times 15^2 \times 12 \end{matrix}$$

$7 \times 5^4 \div 3^3$; $2 \times 90^2 \div 11^3$; $3^6 \times 2^4$; $3^{18} \times 5$;
$5^4 \div 7^4$; $(\frac{3}{11})^6$; $7^3 \times 11^5 \times 13^3 \div 17^2$.

(*Page* 42.)

£9628, 13s. 5,8d. £3829, 6s. 6d. £99834, 11s. 4,8d.

(*Page* 44.)

£429, 14s. 9,67d. £8242, 0s. 11,52d. £2048, 5s. 9,6d.
£6511, 12s. 6,98d. £246010, 3s. 4,8d.

(*Page* 49.)

$\frac{1}{8}$ = ,125 ; $\frac{1}{6561}$ = ,000 152 42 ; $\frac{81}{625}$ = ,1296 ;
$\frac{2401}{16}$ = 150,0625 ; $\frac{1}{1024}$ = ,000 976 562, etc. ;
$\frac{1}{100000}$ = ,000 01 ; $\frac{4}{11}$ = ,3636, etc. ; 781 25 ; 9 ;
$\frac{625}{1089}$ = ,573 92, etc. ;
,014 773 48, etc. = $\frac{1000000}{95443993}$;
7,304 6019, etc. = $\frac{10000}{1369}$;
11 973 037,9186, etc. = $\frac{1000000000000}{83521}$.

(*Page* 51.)

$\frac{1}{3}$; $\frac{1}{25}$; $\frac{1}{28561}$; $\frac{1}{4818900304}$;
$\frac{1}{63759030914653054846432641}$; $\frac{1}{1000000000000000}$;
4; $1\frac{1}{2}$.

(*Page* 52.)

9 256 64 $\frac{1}{4913}$ $\frac{1}{37}$

1 801 152 661 463 $\frac{1}{68\,759\,030\,914\,653\,054\,346\,432\,641}$

$\frac{1}{823543}$ 8192 $878\frac{4243}{6561} = \left(\frac{3}{7}\right)^{-3} \times \left(\frac{3}{7}\right)^{-5}$

$\frac{1}{625}$ $\frac{1}{5\,764\,801}$ $\frac{1}{665\,416\,609\,183\,179\,841}$

(*Page* 53.)

64 6561 2 821 109 907 456 1331 $\frac{1}{12167}$
3 141 126 580 731 587 340 586 460 636 236 874 776 576
$\frac{4096}{531441}$

(*Page* 54.)

$\frac{1}{256}$ $\frac{1}{470\,184\,984\,576}$

$\frac{1}{2\,503\,155\,504\,993\,241\,601\,315\,571\,986\,085\,849}$

$\frac{1}{3\,138\,428\,376\,721}$

61 040 881 526 285 814 362 156 628 321 386 486 455 989 674 569
398 010 574 215 107 679 422 058 885 600 836 061 208 944 572 721

(*Page* 56.)

£6763, 1s. 1·2d. £1228, 0s. 7·7d. £2113, 16s. 11d.
£8517, 3s. 6·2d. £11357, 11s. 9·4d.

(*Page* 59.)

6	13	23	49	105
8	15	30	99	91

(*Page* 61.)

97	943	25 806
131	1047	1 518 595
147	5941	307 158 621
541	13497	1 073 592 714

(*Page* 63.)

125	6 251	703 156
185	6 953	35 473 500
217	9 084	1 111 111 111
1 370	837 095	581 629 374
2 583	347 659	

(*Page* 64.)

17 635	27 176	167 185

(*Page* 66.)

3,162277	13,228756	26,324893	54,781383
6,244998	16,031219	27,874719	308,917465
6,928214	19,773719	30,903074	976,882797
9,643650	20,639767	38,366652	1305,048463
10,677078	21,400934		

(*Page* 68.)

5,91607995	39,92492955	198,74858490
8,83176143	51,16639522	173,00291905
11,57583690	85,77878526	358,99862116
18,86796221		

(*Page* 69.)

2,13822	,146843	,00784462
1,921041	,464359	,76675628718
,370809	1,0001699	2,923576
,9025622	,999829	,9543419
1,657145	18,0642144	

$1\frac{1}{7}$	$\frac{6}{11}$	$\frac{17}{19}$	,9	,23
$\frac{1}{59}$	$1\frac{1}{2}$	$3\frac{1}{3}$	$3\frac{2}{5}$	$8\frac{1}{7}$

(*Page* 70.)

,7071068	,9204467	,5222330
,2582247	1,1902381	,72111026
,8277591	1,0246951	1,3627703
,9128709	,3461094	,9775252
,9176629		

(*Page* 71.)

$\frac{2}{1}, \frac{9}{4}, \frac{38}{17}, \frac{161}{72}, \frac{682}{305}$, &c. $\qquad \frac{5}{1}, \frac{51}{10}, \frac{515}{101}, \frac{5201}{1020}$, &c.

$\frac{5}{1}, \frac{6}{1}, \frac{11}{2}, \frac{39}{7}, \ldots \frac{16063}{2885}$, &c. $\qquad \frac{2}{1}, \frac{3}{1}, \frac{14}{5}, \frac{17}{6}, \frac{82}{29}, \frac{99}{35}$, &c.

$\frac{9}{1}, \frac{163}{18}, \frac{2943}{325}$, &c. $\qquad \frac{11}{1}, \frac{12}{1}, \frac{275}{23}, \frac{287}{24}, \frac{6589}{551}$, &c.

$\frac{12}{1}, \frac{289}{24}, \frac{6948}{577}$, &c. $\qquad \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{17}{26}, \frac{19}{29}, \frac{36}{55}, \frac{91}{139}$, &c.

$\frac{1}{1}, \frac{2}{3}, \frac{5}{7}, \frac{12}{17}, \frac{29}{41}$, &c. $\qquad \frac{1}{1}, \frac{7}{6}, \frac{8}{7}, \frac{231}{202}, \frac{239}{209}, \frac{1665}{1456}, \frac{3569}{3121}$, &c.

(*Page* 74.)

33	111	274	862
41	145	583	905

(*Page* 77.)

3 215 578 176 59 547 442 625 231 815 006 207
440 589 476 776 357 8 944 897 170 224 289

(*Page* 78.)

24	84,2311	21954
34	151,9204	76101
47	1307,0000	5,3565337
62	8573,	80079

(*Page* 81.)

1,2599021051	and	1,5874010502
2,1544346901	and	4,6415888321
4,1911087263	and	17,5653923682
,9226357936	and	,8512568074
,3206410577	and	,1028102145
,1945925378	and	,0378662530
1,4422495706	and	2,0800837895

11,1190045752	and	123,6521861345
7,8515562954	and	61,6469362704
,7179054354	and	,5153882140
,7572886307	and	,5734860711
1,0011653094	and	1,0023319754
,0791715773	and	,0062681359

(*Page* 82.)

$\frac{30}{64}$	$\frac{17475}{27000}$	$\frac{15}{27}$	$\frac{9482}{10648}$
$\frac{172}{216}$	$\frac{167629}{456533}$	$\frac{380}{1000}$	$\frac{59251}{59319}$
$\frac{6439}{9261}$	$\frac{7526}{8000}$	$\frac{3285}{3375}$	$\frac{10053527}{9129329}$

(*Page* 86.)

16,656857,	277,450896,	and	4631,469991
4,114422,	16,928467,	and	69,650855
1,717127,	2,948525,	and	5,062992
,924223,	,854188,	and	,789460
,369919,	,136839,	and	,050620
4,394678,	19,313206,	and	84,875359
3,18137,	10,121116,	and	32,199019
1,281035,	1,641051,	and	2,102206
,653817,	,427476,	and	,279492
,239221,	,057227,	and	,013690

(*Page* 88.)

2,058924	1,760968	,4356025
3,001174	1,105889	,7418806
3,093367	1,053559	1,0019565

(*Page* 91.)

3,3019272	40,8232162	,0223351
2,6918012	2027,717281	,000041526
62,8970793	4,7692355	,00040952
1,5409891	2,4791914	
6,4351133	,0102877	

,29907	,3015113	,4805411	,7198244
2,695882	17,604734	1,090546	

(*Page* 91.)

5^{2}	$13^{\frac{19}{15}}$	$5^{\frac{15}{4}}$	$6^{-\frac{2}{3}}$
$9^{\frac{6}{5}}$	6^{-1}	$17^{\frac{2}{7}}$	$3^{\frac{8}{5}}$
7^{-5}	$11^{\frac{3}{5}}$	$8^{\frac{103}{132}}$	13^{-1}
$7^{2\frac{1}{3}}$	$4^{\frac{46}{35}}$	$23^{\frac{12}{35}}$	$23^{\frac{4}{7}}$

(*Page* 100.)

4,247928	1,920506	2,959481

(*Page* 104.)

,47712 12547	,84509 80400
1,04139 26852	1,11394 33523

(*Page* 111.)

2,935 0032	6,472 5663	0,915 0197
3,640 4814	1,991 9434	0,993 0260
3,492 2015	0,871 0542	8,954 2425
4,473 6475	2,707 0419	3,981 5785
2,922 9330	1,795 4281	3,855 1071

(*Page* 112.)

9,477 1213	9,922 9485	6,795 8800
8,672 0979	8,999 1305	4,642 4645
7,857 9353	5,754 3483	

(*Page* 115.)

0,497 1499	1,138 6984	0,882 1657
9,637 7843	2,690 8776	9,909 9494
2,562 5809	4,582 9001	3,631 0903
0,922 2542	7,821 6284	4,030 9836
1,762 0059	8,270 4321	3,339 7441

(*Page* 117.)

615,66	1,7332	584,32
7014,8	,2094901	71,783
3,8936	,0046083	,087708

11,1190045752	and	123,6521861345
7,8515562954	and	61,6469362704
,7179054354	and	,5153882140
,7572886307	and	,5734860711
1,0011653094	and	1,0023319754
,0791715773	and	,0062681359

(*Page* 82.)

$\frac{39}{64}$	$\frac{17475}{27000}$	$\frac{15}{27}$	$\frac{9482}{10648}$
$\frac{172}{216}$	$\frac{167629}{456533}$	$\frac{380}{1000}$	$\frac{59251}{59319}$
$\frac{6439}{9261}$	$\frac{7526}{8000}$	$\frac{3285}{3375}$	$\frac{10053527}{9129329}$

(*Page* 86.)

16,656857,	277,450896,	and	4631,469991
4,114422,	16,928467,	and	69,650855
1,717127,	2,948525,	and	5,062992
,924223,	,854188,	and	,789460
,369919,	,136839,	and	,050620
4,394678,	19,313206,	and	84,875359
3,18137,	10,121116,	and	32,199019
1,281035,	1,641051,	and	2,102206
,653817,	,427476,	and	,279492
,239221,	,057227,	and	,013690

(*Page* 88.)

2,058924	1,760968	,4356025
3,001174	1,105889	,7418806
3,093367	1,053559	1,0019565

(*Page* 91.)

3,3019272	40,8232162	,0223351
2,6918012	2027,717281	,000041526
62,8970793	4,7692355	,00040952
1,5409891	2,4791914	
6,4351133	,0102877	

,29907	,3015113	,4805411	,7198244
2,695882	17,604734	1,090546	

(*Page* 91.)

5^{2}	$13^{\frac{19}{15}}$	$5^{\frac{15}{4}}$	$6^{-\frac{2}{3}}$
$9^{\frac{6}{5}}$	6^{-1}	$17^{\frac{2}{7}}$	$3^{\frac{8}{5}}$
7^{-5}	$11^{\frac{3}{5}}$	$8^{\frac{103}{132}}$	13^{-1}
$7^{2\frac{1}{3}}$	$4^{\frac{46}{35}}$	$23^{\frac{12}{35}}$	$23^{\frac{4}{7}}$

(*Page* 100.)

4,247928	1,920506	2,959481

(*Page* 104.)

,47712 12547	,84509 80400
1,04139 26852	1,11394 33523

(*Page* 111.)

2,935 0032	6,472 5663	0,915 0197
3,640 4814	1,991 9434	0,993 0260
3,492 2015	0,871 0542	8,954 2425
4,473 6475	2,707 0419	3,981 5785
2,922 9330	1,795 4281	3,855 1071

(*Page* 112.)

9,477 1213	9,922 9485	6,795 8800
8,672 0979	8,999 1305	4,642 4645
7,857 9353	5,754 3483	

(*Page* 115.)

0,497 1499	1,138 6984	0,882 1657
9,637 7843	2,690 8776	9,909 9494
2,562 5809	4,582 9001	3,631 0903
0,922 2542	7,821 6284	4,030 9836
1,762 0059	8,270 4321	3,339 7441

(*Page* 117.)

615,66	1,7332	584,32
7014,8	,2094901	71,783
3,8936	,0046083	,087708

(*Page* 118.)

2904,1073
40,063565
,03810678
,001157375
2,107438

164,621 72
672848,46
,481 6529
,033 00934
6,016 757

(*Page* 123.)

84,165, or 84,16328 ; 2695,6 lb. ; ,4912938
83729,5 gallons ; 171 0260,8 gallons ;
164 603 pounds ; £13, 1s. 1,5d ; 2 521,5 tons ;
£146, 16s. 1d. ; 67,04426 ; £5368, 12s. 6d. ;
£8324, 9s. 6,7d. ; £32360, 4s. 7,2d.

(*Page* 125.)

13,12532 grains ; 8,263 912 grains ; £2, 3s. 8d. ;
£4493, 2s. 8,2d. ; 59′...8″,331, or 1,095163 ;
£3, 0s. 7d.

(*Page* 126.)

2074,17 tons ; 4,009766 gallons ; 23312,53 pounds ;
,000 015 471 inch ; ,13872 inch.

(*Page* 127.)

6,86814 gallons ; 1°...22′...20″,434 ; $1^{h}\ 51^{m}\ 23^{s}$,15 ;
,0927925 of a grain ; 43,0266 grains.

(*Page* 129.)

A.'s share = £878, 13s. 10,2d. ; B.'s = £128, 12s. 10,3d. ; C.'s = £502, 7s. ,2d. ; D.'s = £573, 6s. 9,4d. ; E.'s = £9, 17s. 3,7d. ; F.'s = £34, 15s. 9,5d. ; G.'s = £172, 13s. 8,7d. ; H.'s = £77, 8s. 10d. ; I.'s = £326, 19s. 10,7d. ; K.'s = 17s. 5,2d. ; L.'s = £1, 14s. 5,9d. ; M.'s = £2, 17s. 1,4d. ; N.'s = £13, 12s. 11,7d. ; O.'s = 10s. 9d. ; P.'s = 4s. 9,8d. ; and Q.'s = £45, 4s. 7,2d.

A.'s share = £4128, 18s. 9,1d.; B.'s = £3254, 15s. 7,7d.; C.'s £2267, 14s. 7,4d.; D.'s = £1227, 17s. 2,2d.; E.'s = £306, 11s. 6,2d.; F.'s = £871 11s. 4d.; G.'s = £112, 7s. 6,2d.; H.'s = £1580, 0s. 1,4d.; I.'s = £40, 18s. 2,6d.; K.'s = £426, 4s. 0,7d.; L.'s = £4820, 5s. 8,9d.; M.'s = £3737, 8s. 7,7d.; N.'s = £1031, 1s. 4,6d.; O.'s = £2127, 18s. 1,7d.; P.'s = £1225, 15s. 8,9d.; Q.'s = £2943, 10s. 10,8d.; R.'s = £6418, 18s. 8,9d.; S.'s = £396, 3s. 4,1d.; and T.'s = £3805, 15s. 9,4d.

(*Page* 130.)

3944,9245, or 3944,92; 96 185 300 000, or 96 185 464 000; 143,41687, or 143,41711.

(*Page* 131.)

,644 771	,267 605 6	,000 000 001 377 36
,999 9407	,000 000 013 411 72	,003 745 056

(*Page* 132.)

£8487, 17s. 5,76d.; £16 446; £17 146, 12s. 5,76d., £17 529, 3s. 9,6d., and £17 797, 11s. 2,2d.;
£689 804 904 000 000 000 000 000;
£151, 5s. 2,2d., £173, 3s. 4,2d., £197, 19s. 11,55d., and £226, 1s. 9,7d.

,054 4277	,011 6533	1,037 9905
,054 1815	,044 5913	21,101 973
,028 0375	,975 6817	15773,591

(*Page* 133.)

£3791, 9s. 6,7d., £3279, 18s. 11,5d., £2841, 7s. 4,3d., and £2464, 15s. 8,2d.
£,000 000 091 969 4, and £,000 000 000 023 538.

(*Page* 134.)

4,924 812	1,009 865	1,379 730
,835 5264	,778 7385	1,348 006
,092 8725	,096 4415	1,320 469
2,544 039	1,414 214	1,296 84
1,464 592	1,442 249	1,276 518
,408 472	1,414 214	1,258 925

5,008 per cent. 3,5265 per cent.

(*Page* 136.)

3,109 211	,008 088 002	6,573 06
5685,321	,225 383 0	9508,166
8,982 416	,043 991 6	2,631 271
2,860 484	,334 331 5	
,881 2694	1,067 738	

23,212 years; 23,45 years, 17,673 years, 14,24 years, 11,896 years, and 10,245 years.

(*Page* 147.)

Binary scale, 11 110 100 001 001 000 000;
Ternary, 1 212 210 202 001; Quaternary, 3 310 021 000;
Quinary, 224 000 000; Senary, 33 233 344;
Septenary, 11 333 311; Octenary, 3 641 100;
Nonary, 1 783 661.

(*Page* 148.)

Binary, ,100 010 011 101, repeated; ,1001, repeated; ,101 111 010 01 repeated.
Ternary, ,1121, repeated; ,1210, repeated; ,201 221 211 02, repeated.
Quaternary, ,202 131, repeated; ,21, repeated; ,233 103 132 21, repeated.
Quinary, ,2321; ,3; ,332 143 424 031 123 010 204 1, repeated.
Senary, ,312 150 243 405, repeated; ,3, repeated; ,423 352 511 45, repeated.

Septenary, ‚352 456 314 210, repeated ; ‚4125, repeated ; ‚511 343 646 041 553 230 206 2, repeated.

Octenary, ‚4235, repeated ; ‚4631, repeated ; ‚572 336 467 51, repeated.

Nonary, ‚475, repeated ; ‚53, repeated ; ‚657 738 185 42, repeated.

(*Page* 150.)

43‚15410425, &c., 36‚21631203, &c., and 30‚28278431, &c.; 12‚3011414, &c., 11‚335550452, &c.

(*Page* 152.)

39 273 square feet, 3 square inches, and 9‚3 square lines ;
743 square feet, 4 square inches, and 4‚5 square lines ;
10254 cubic feet, 6 cubic inches, and ‚125 of a cubic line ;
53 square feet, 5 square inches, and 8 square lines ;
1559 cubic feet, 9 cubic inches, 3‚6 cubic lines.

(*Page* 153.)

11 feet, and $\frac{10}{23}$ of a line ; 1 foot, 3 inches, 7 lines, and 3...3.

(*Page* 154.)

11 feet 11...2...1...10...8 ; 10‚4...1...6...6...10...2, &c.

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